Completeness of Kozen's Axiomatization for the Modal μ-Calculus: A Simple Proof

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Abstract

The modal μ-calculus, introduced by Dexter Kozen, is an extension of modal logic with fixpoint operators. Its axiomatization, Koz, was introduced at the same time and is an extension of the minimal modal logic K with the so-called Park fixpoint induction principle. It took more than a decade for the completeness of Koz to be proven, finally achieved by Igor Walukiewicz. However, his proof is fairly involved.

In this article, we present an improved proof for the completeness of Koz which, although similar to the original, is simpler and easier to understand.

Keywords: The modal μ-calculus, completeness, parity games, parity automata.

1 Introduction

The *modal μ-calculus* originated with Scott and De Bakker [4] and was further developed by Dexter Kozen [8] into the main version currently used. It is used to describe and verify properties of labeled transition systems (Kripke models). Many modal and temporal logics can be encoded into the modal μ-calculus, including CTL^* and its widely used fragments – the linear temporal logic LTL and the computational tree logic CTL. The modal μ-calculus also provides one of the strongest examples of the connections between modal and temporal logics, automata theory and game theory (for example, see [6]). As such, the modal μ-calculus is a very active research area in both theoretical and practical computer science. We refer the reader to Bradfield and Stirling's tutorial article [3] for a thorough introduction to this formal system.

The difference between the modal μ-calculus and modal logic is that the former has the *least fixpoint operator* μ and the *greatest fixpoint operator* ν which represent the least and greatest fixpoint solution to the equation $\alpha(x) = x$, where $\alpha(x)$ is a monotonic function mapping some power set of possible worlds into itself.[1] In Kozen's initial work [8], he proposed an axiomatization Koz, which was an extension of the minimal modal logic K with a further axiom and inference rule – the so-called Park fixpoint induction principle:

$$\frac{}{\alpha(\mu x.\alpha(x)) \vdash \mu x.\alpha(x)}\ (\mathsf{Prefix}) \qquad \frac{\alpha(\beta) \vdash \beta}{\mu x.\alpha(x) \vdash \beta}\ (\mathsf{Ind})$$

The system Koz is very simple and natural; nevertheless, Kozen himself could not prove completeness for the full language, but only for the negations of formulas of a special kind called the *aconjunctive formula*. Completeness for the full language turned out to be a knotty problem and remained open for more than a decade. Finally, Walukiewicz [15] solved this problem, but his proof is quite involved.[2]

The aim of this article is to provide an improved proof that is easier to understand. First, we outline Walukiewicz's proof and explain its difficulties, and then present our improvement.

The completeness theorem considered here is sometimes called weak completeness and requires that the validity follows the provability; that is:

(a) For any formula φ, if φ is not satisfiable, then $\sim\varphi$ is provable in Koz.

[1] In the modal μ-calculus, the term *state* is preferred to *possible world* since it originated in the area of verification of computer systems. However, we do not use this terminology since it is reserved for *state of automata* in this article.

[2] The difficulties of the proof have been pointed out, e.g., see [3, 1, 2, 14, 9]

Here, $\sim \varphi$ denotes the negation of φ. Note that strong completeness cannot be applied to the modal μ-calculus since it lacks compactness. The first step of the proof is based on the results of Janin and Walukiewicz [7], in which they introduced the class of formulas called *automaton normal form*,[3] and showed the following two theorems:

(b) For any formula φ, we can construct an automaton normal form $\mathsf{anf}(\varphi)$ which is semantically equivalent to φ.

(c) For any automaton normal form $\widehat{\varphi}$, if $\widehat{\varphi}$ is not satisfiable, then $\sim\widehat{\varphi}$ is provable in Koz; that is, Koz is complete for the negations of the automaton normal form.

The above theorems lead to the following Claim (d) for proving:

(d) For any formula φ, there exists a semantically equivalent automaton normal form $\widehat{\varphi}$ such that $\varphi \to \widehat{\varphi}$ is provable in Koz.

Indeed, for any unsatisfiable formula φ, Claim (d) tells us that $\sim\widehat{\varphi} \to \sim\varphi$ is provable; on the other hand, from Theorem (c) we obtain that $\sim\widehat{\varphi}$ is provable; therefore $\sim\varphi$ is provable in Koz as required. Hence, our target (a) is reduced to Claim (d).

Another important tool is the concept of a *tableau*, which is a tree structure that is labeled by some subformulas of the primary formula φ and is related to the satisfiability problem for φ. Niwinski and Walukiewicz [11] introduced a game played by two adversaries on a tableau (called *tableau games* in this article) and, by analyzing these games, showed that:

(e) For any unsatisfiable formula φ, there exists a structure called the *refutation* for φ which is a substructure of tableau.

Importantly, a refutation for φ is very similar to a proof diagram for φ; roughly speaking, the difference between them is that the former can have infinite branches while the latter can not. Walukiewicz shows that if the refutation for φ satisfies a special *thin* condition, it can be transformed into a proof diagram for φ. In other words,

(f) For any unsatisfiable formula φ such that there exists a thin refutation for φ, $\sim\varphi$ is provable in Koz.

Note that Claim (f) is a slight generalization of the completeness for the negations of the aconjunctive formula in the sense that the refutation for an unsatisfiable aconjunctive formula is always thin, and Claim (f) can be shown by the same method as Kozen's original argument.

The proof is based on confirming Claim (d) by induction on the length of φ, using (b) and (f). The hardest step of induction is the case $\varphi = \mu x.\alpha(x)$. Suppose $\varphi = \mu x.\alpha(x)$ and that we could assume, by inductive hypothesis, $\alpha(x) \to \widehat{\alpha}(x)$ is provable in Koz where $\widehat{\alpha}(x)$ is an automaton normal form equivalent to $\alpha(x)$. For the inductive step, we want to discover an automaton normal form $\widehat{\varphi}$ equivalent to $\mu x.\alpha(x)$ such that $\mu x.\alpha(x) \to \widehat{\varphi}$ is provable. Note that since $\alpha(x) \to \widehat{\alpha}(x)$ is provable, $\mu x.\alpha(x) \to \mu x.\widehat{\alpha}(x)$ is also provable. Furthermore, $\mu x.\alpha(x)$ and $\mu x.\widehat{\alpha}(x)$ are equivalent to each other. Set $\widehat{\varphi} := \mathsf{anf}(\mu x.\widehat{\alpha}(x))$. Then, it is sufficient to show that $\mu x.\widehat{\alpha}(x) \to \widehat{\varphi}$ is provable, and thus, from the induction rule (Ind), $\widehat{\alpha}(\widehat{\varphi}) \to \widehat{\varphi}$ is provable. To show this, Walukiewicz developed a new utility called *tableau consequence*, which is a binary relation on the tableau and is characterized using game theoretical notations. The following two facts were then shown:

(g) Let $\widehat{\alpha}(x)$ and $\widehat{\varphi}$ be formulas denoted above. Then the tableau for $\widehat{\varphi}$ is a consequence of the tableau for $\widehat{\alpha}(\widehat{\varphi})$.

(h) For any automaton normal forms $\widehat{\beta}(y)$ and $\widehat{\psi}$, if the tableau for $\widehat{\psi}$ is a consequence of the tableau for $\widehat{\beta}(\widehat{\psi})$, then we can construct a thin refutation for $\sim(\widehat{\beta}(\widehat{\psi}) \to \widehat{\psi})$.[4]

[3] In the original article [7], this class of formulas was called the *disjunctive formula*; however, the term *automaton normal form* is the currently used terminology, to the author's knowledge.

[4] More precisely, this assertion must be stated more generally to be applicable in other cases of an inductive step, see Lemma 6.8.

The real difficulty appeared when proving Claim (g). To establish this claim, Walukiewicz introduced complicated functions across some tableaux and analyzed the properties of these functions very carefully. Finally, Claims (f), (g) and (h) together immediately establish that $\widehat{\alpha}(\widehat{\varphi}) \to \widehat{\varphi}$ is provable in Koz. Thus, he obtained a proof for Claim (d), confirming completeness.

This article's main contribution is the simplification of the proof of Claim (g). For this purpose, we will introduce a new tableau-like structure called a *wide tableau* and provide a more suitable re-formulation of the concept of tableau consequence to prove Claim (g). This re-formulation will be defined similarly to the concept of *bisimulation* (instead of the game theoretical notations), which is one of the most fundamental and standard notions in the model theory of modal and its extensional logics. Consequently, although our proof of completeness does not include any innovative concepts, it is far more concise than the original proof.

The author hopes that the method given in this article may assist investigation of the modal μ-calculus and related topics.

1.1 Outline of the article

The remainder of this article is organized as follows: in the following subsection 1.2, we will define some terminologies used within the article. Section 2 gives basic definitions of the syntax and semantics of the modal μ-calculus. Section 3 and 4 introduce well known results concerning parity automata and parity games, respectively. Section 5 contains the principle part of this article – the proof of Claim (g). For this proof, Claim (b) and the techniques used for proving (b) are fundamental. Therefore, we recount the argument of Janin and Walukiewicz [7] in detail. In Section 6, we prove the completeness of Koz by showing Claim (d).

1.2 Notation

Sets: Let X be an arbitrary set. The *cardinality of* X is denoted $|X|$. The *power set of* X is denoted $\mathcal{P}(X)$. ω denotes the set of natural numbers.

Sequences: A finite sequence over some set X is a function $\pi : \{1, \dots, n\} \to X$ where $1 \leq n$. An infinite sequence over X is a function $\pi : \omega \setminus \{0\} \to X$. Here, a sequence can refer to either a finite or infinite sequence. The length of a sequence π is denoted $|\pi|$. Let π be a sequence over X. The set of $x \in X$ which appears infinitely often in π is denoted $\mathsf{Inf}(\pi)$. We denote the n-th element in π by $\pi[n]$ and the fragment of π from the n-th element to the m-th element by $\pi[n, m]$. For example, if $\pi = \mathsf{aabbcddd}$, then $\pi[5] = \mathsf{c}$ and $\pi[2, 6] = \mathsf{abbcd}$. Note that when π is a finite non-empty sequence, $\pi[|\pi|]$ denotes the tail of π.

Alphabets: Suppose that Σ is a non-empty finite set. Then we may call Σ an *alphabet* and its element $v \in \Sigma$ a *letter*. We denote the set of finite sequences over Σ by Σ^*, the set of non-empty finite sequences over Σ by Σ^+, and the set of infinite sequences over Σ by Σ^ω. As usual, we call an element of Σ^* a *word*, an element of Σ^ω an *ω-word*, a set of finite words $\mathcal{L} \subseteq \Sigma^*$ a *language* and, a set of ω-words $\mathcal{L}' \subseteq \Sigma^\omega$ an *ω-language*. The notion of the *factor* on words is defined as usual: for two words $u, v \in \Sigma^* \cup \Sigma^\omega$, u is a factor of v if $v = xuy$ for some $x, y \in \Sigma^* \cup \Sigma^\omega$.

Graphs: In this article, the term *graph* refers to a directed graph. That is, a graph is a pair $\mathcal{G} = (V, E)$ where V is an arbitrary set of *vertices* and E is an arbitrary binary relation over V, i.e., $E \subseteq V \times V$. A vertex u is said to be an E-successor (or simply a successor) of a vertex v in $\mathcal{G}$ if $(v, u) \in E$. For any vertex v, we denote the set of all E-successors of v by $E(v)$. The sequence $\pi \in V^* \cup V^\omega$ is called an *E-sequence* if $\pi[n + 1] \in E(\pi[n])$ for any $n < |\pi|$. E^* denotes the reflexive transitive closure of E and E^+ denotes the transitive closure of E.

Trees: The term *tree* is used to mean a *rooted direct tree*. More precisely, a tree is a triple $\mathcal{T} = (T, C, r)$ where T is a set of *nodes*, $r \in T$ is a *root* of the tree and, C is a *child relation*, i.e., $C \subseteq T \times T$ such that for any $t \in T \setminus \{r\}$, there is exactly one C-sequence starting at r and ending at t. As usual, we say that u is a child of t (or t is a parent of u) if $(t, u) \in C$. A node $t \in T$ is a *leaf* if $C(t) = \emptyset$. A *branch* of $\mathcal{T}$ is either a finite C-sequence starting at r and ending at a leaf or an infinite C-sequence starting at r.

Unwinding: Let $\mathcal{G} = (V, E)$ be a graph. An *unwinding* of $\mathcal{G}$ on $v \in V$ is the tree structure $\mathsf{UNW}_v(\mathcal{G}) = (T, C, r)$ where:

- T consists of all finite non-empty E-sequences that start at v,
- $(\pi, \pi') \in C$ if and only if; $|\pi| + 1 = |\pi'|$, $\pi = \pi'[1, |\pi|]$ and $(\pi[|\pi|], \pi'[|\pi'|]) \in E$, and
- $r := v$.

This concept can be extended naturally into a graph with some additional relations or functions. For example, let $\mathcal{S} = (V, E, f)$ be a structure where $\mathcal{G} = (V, E)$ is a graph and f is a function with domain V. Then we define $\mathsf{UNW}_v(\mathcal{S}) := (\mathsf{UNW}_v(\mathcal{G}), f')$ as $f'(\pi) := f(\pi[|\pi|])$ for any $\pi \in V^+$. Note that we use the same symbol f instead of f' in $\mathsf{UNW}_v(\mathcal{S})$ if there is no danger of confusion.

Functions: Let f be a function from some set X to some set Y. We define the new function $\vec{f}$ from $X^+ \cup X^\omega$ to $Y^+ \cup Y^\omega$ as:

$$\vec{f}(\pi) := f(\pi[1])f(\pi[2])\cdots$$

where $\pi \in X^+ \cup X^\omega$. It is obvious that for any $\pi \in X^+ \cup X^\omega$, we have $|\pi| = |\vec{f}(\pi)|$.

2 The modal μ-calculus

We will now introduce the syntax, semantics and axiomatization Koz of the modal μ-calculus, and then present some additional concepts and results for use in the following sections.

2.1 Syntax

Definition 2.1 (Formula). Let $\mathsf{Prop} = \{p, q, r, x, y, z, \dots\}$ be an infinite countable set of *propositional variables*. Then the collection of the *modal μ-formulas* is defined as follows:

$$\varphi ::= (\top), (\bot), (p) \mid (\neg p) \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid (\Diamond \varphi) \mid (\Box \varphi) \mid (\mu x.\varphi) \mid (\nu x.\varphi)$$

where $p, x \in \mathsf{Prop}$. Moreover, for formulas of the form $(\sigma x.\varphi)$ with $\sigma \in \{\mu, \nu\}$, we require that each occurrence of x in φ is positive; that is, $\neg x$ is not a subformula of φ. Henceforth in this article, we will use σ to denote μ or ν. A formula of the form p or $\neg p$ for $p \in \mathsf{Prop}$, $\top$ and $\bot$ is called *literal.* We use the term Lit to refer to the set of all literals, i.e., $\mathsf{Lit} := \{p, \neg p, \bot, \top \mid p \in \mathsf{Prop}\}$. We call μ and ν *the least fixpoint operator* and *the greatest fixpoint operator*, respectively.

Remark 2.2. In Definition 2.1, we confined the formula to a *negation normal form*; that is, the negation symbol may only be applied to propositional variables. However, this restriction can be inconvenient, and so we extend the concept of the negation to an arbitrary formula φ (denoted by $\sim\varphi$) inductively as follows:

- $\sim\top := \bot$, $\sim\bot := \top$.
- $\sim p := \neg p$, $\sim\neg p := p$ for $p \in \mathsf{Prop}$.
- $\sim(\varphi \vee \psi) := ((\sim\varphi) \wedge (\sim\psi))$, $\sim(\varphi \wedge \psi) := ((\sim\varphi) \vee (\sim\psi))$.
- $\sim(\Diamond\varphi) := (\Box(\sim\varphi))$, $\sim(\Box\varphi) := (\Diamond(\sim\varphi))$.
- $\sim(\mu x.\varphi(x)) := (\nu x.(\sim\varphi(\neg x)))$, $\sim(\nu x.\varphi(x)) := (\mu x.(\sim\varphi(\neg x)))$.

We introduce *implication* $(\varphi \to \psi)$ as $((\sim\varphi) \vee \psi)$ and *equivalence* $(\varphi \leftrightarrow \psi)$ as $((\varphi \to \psi) \wedge (\psi \to \varphi))$ as per the usual notation. To minimize the use of parentheses, we assume the following precedence of operators from highest to lowest: $\neg$, $\sim$, $\Diamond$, $\Box$, σx, $\vee$, $\wedge$, $\to$ and $\leftrightarrow$. Moreover, we often abbreviate the outermost parentheses. For example, we write $\Diamond p \to q$ for $((\Diamond p) \to q)$ but not for $(\Diamond(p \to q))$.

As fixpoint operators μ and ν can be viewed as quantifiers, we use the standard terminology and notations for quantifiers. We denote the set of all propositional variables appearing free in φ by $\mathsf{Free}(\varphi)$, and those appearing bound by $\mathsf{Bound}(\varphi)$. If ψ is a subformula of φ, we write $\psi \leq \varphi$. We write $\psi < \varphi$ when ψ is a proper subformula. $\mathsf{Sub}(\varphi)$ is the set of all subformulas of φ and $\mathsf{Lit}(\varphi)$ denotes the set of all literals which are subformulas of φ. Let $\varphi(x)$ and ψ be two formulas. The *substitution* of all free appearances of x with ψ into φ is denoted $\varphi(x)[x/\psi]$ or sometimes simply $\varphi(\psi)$. As with predicate logic, we prohibit substitution when a new binding relation will occur by that substitution.

The following two definitions regarding formulas will be used frequently in the remainder of the article.

Definition 2.3 (**Well-named formula).** The set of *well-named formulas* WNF is defined inductively as follows:

1. $\mathsf{Lit} \subseteq \mathsf{WNF}$.
2. Let $\alpha, \beta \in \mathsf{WNF}$ where $\mathsf{Bound}(\alpha) \cap \mathsf{Free}(\beta) = \emptyset$ and $\mathsf{Free}(\alpha) \cap \mathsf{Bound}(\beta) = \emptyset$. Then $\alpha \vee \beta, \alpha \wedge \beta \in \mathsf{WNF}$.
3. Let $\alpha \in \mathsf{WNF}$. Then $\Diamond\alpha, \Box\alpha \in \mathsf{WNF}$.
4. Let $\alpha(x) \in \mathsf{WNF}$ where $x \in \mathsf{Free}(\alpha(x))$ occurs only positively, moreover, x is in the scope of some modal operators. Then $\sigma x_1. \ldots \sigma x_k.\alpha(x_1, \ldots, x_k) \in \mathsf{WNF}$ where $\alpha(x) = \alpha(x_1, \ldots, x_k)[x_1/x, \ldots, x_k/x]$, $x \notin \mathsf{Sub}(\alpha(x_1, \ldots, x_k))$ and $x_1, \ldots, x_k \notin \mathsf{Sub}(\alpha(x))$.

The formula $\sigma x_1. \ldots \sigma x_k.\alpha(x_1, \ldots, x_k)$ which is mentioned above clause 4 is sometimes abbreviated $\sigma\vec{x}.\alpha(\vec{x})$. If φ is well-named and x is bounded in φ, then there is exactly one subformula which binds x; this formula is denoted $\sigma_x x.\varphi_x(x)$.

Definition 2.4 (**Alternation depth).** Given a formula φ,

1. Let $\preceq^-_\varphi$ be a binary relation on $\mathsf{Bound}(\varphi)$ such that $x \preceq^-_\varphi y$ if and only if $x \in \mathsf{Free}(\varphi_y(y))$. The *dependency order* $\preceq_\varphi$ is defined as the transitive closure of $\preceq^-_\varphi$.
2. A sequence $\langle x_1, x_2, \ldots, x_K\rangle \in \mathsf{Bound}(\varphi)^+$ is said to be an alternating chain if:

$$x_1 \preceq^-_\varphi x_2 \preceq^-_\varphi \cdots \preceq^-_\varphi x_K$$

   and $\sigma_{x_k} \neq \sigma_{x_{k+1}}$ for every $k \in \omega$ such that $1 \leq k \leq K-1$. The *alternation depth* of φ (denoted $\mathsf{alt}(\varphi)$) is the maximal length of alternating chains in φ. That is, the alternation depth of φ is the maximal number of alternations between μ- and ν-operators in φ.

Example 2.5. For a formula $\varphi = \mu x.\nu y.(\Diamond x \vee (\mu z.(\Diamond z \wedge \Box y)))$, we have $\mathsf{alt}(\varphi) = 3$ since $x \preceq^-_\varphi y \preceq^-_\varphi z$ with $\sigma_x \neq \sigma_y$ and $\sigma_y \neq \sigma_z$. Note that although $x \notin \mathsf{Free}(\varphi_z(z))$, we have $x \preceq_\varphi z$.

2.2 Semantics

Definition 2.6 (**Kripke model).** A *Kripke model* for the modal μ-calculus is a structure $\mathcal{S} = (S, R, \lambda)$ such that:

- $S = \{s, t, u, \ldots\}$ is a non-empty set of *possible worlds*.
- R is a binary relation over S called the *accessibility relation*.
- $\lambda : \mathsf{Prop} \to \mathcal{P}(S)$ is a *valuation*.

Definition 2.7 (**Denotation).** Let $\mathcal{S} = (S, R, \lambda)$ be a Kripke model and let x be a propositional variable. Then for any set of possible worlds $T \in \mathcal{P}(S)$, we can define a new valuation $\lambda[x \mapsto T]$ on S as follows:

$$\lambda[x \mapsto T](p) := \begin{cases} T & \text{if } p = x, \\ \lambda(p) & \text{otherwise.} \end{cases}$$

Moreover, $\mathcal{S}[x \mapsto T]$ denotes the Kripke model $(S, R, \lambda[x \mapsto T])$. A *denotation* $[\![\varphi]\!]_\mathcal{S} \in \mathcal{P}(S)$ of a formula φ on $\mathcal{S}$ is defined inductively on the structure of φ as follows:

- $[\![\bot]\!]_\mathcal{S} := \emptyset$ and $[\![\top]\!]_\mathcal{S} := S$.
- $[\![p]\!]_\mathcal{S} := \lambda(p)$ and $[\![\neg p]\!]_\mathcal{S} := S \setminus \lambda(p)$ for any $p \in \mathsf{Prop}$.
- $[\![\varphi \vee \psi]\!]_\mathcal{S} := [\![\varphi]\!]_\mathcal{S} \cup [\![\psi]\!]_\mathcal{S}$ and $[\![\varphi \wedge \psi]\!]_\mathcal{S} := [\![\varphi]\!]_\mathcal{S} \cap [\![\psi]\!]_\mathcal{S}$.

- $[\![\Diamond\varphi]\!]_{\mathcal{S}} := \{s \mid \exists t \in S, (s,t) \in R \wedge t \in [\![\varphi]\!]_{\mathcal{S}}\}$.
- $[\![\Box\varphi]\!]_{\mathcal{S}} := \{s \mid \forall t \in S, (s,t) \in R \Longrightarrow t \in [\![\varphi]\!]_{\mathcal{S}}\}$.
- $[\![\mu x.\varphi(x)]\!]_{\mathcal{S}} := \bigcap\{T \in \mathcal{P}(S) \mid [\![\varphi(x)]\!]_{\mathcal{S}[x\mapsto T]} \subseteq T\}$.
- $[\![\nu x.\varphi(x)]\!]_{\mathcal{S}} := \bigcup\{T \in \mathcal{P}(S) \mid T \subseteq [\![\varphi(x)]\!]_{\mathcal{S}[x\mapsto T]}\}$.

In accordance with the usual terminology, we say that a formula φ is *true* or *satisfied* at a possible world $s \in S$ (denoted $\mathcal{S}, s \models \varphi$) if $s \in [\![\varphi]\!]_{\mathcal{S}}$. A formula φ is *valid* (denoted $\models \varphi$) if φ is true at every world in any model.

Example 2.8. Let $\mathcal{S} = (S, R, \lambda)$ be a Kripke model. A formula $\varphi(x)$ such that $x \in \mathsf{Free}(\varphi(x))$ can be naturally seen as the following function:

$$\begin{array}{ccc} \mathcal{P}(S) & \longrightarrow & \mathcal{P}(S) \\ \in & & \in \\ T & \longmapsto & [\![\varphi(x)]\!]_{\mathcal{S}[x\mapsto T]}. \end{array}$$

This function is *monotone* if x is positive in $\varphi(x)$. Thus, by the Knaster-Tarski Theorem [13], $[\![\mu x.\varphi(x)]\!]_{\mathcal{S}}$ and $[\![\nu x.\varphi(x)]\!]_{\mathcal{S}}$ are the least and greatest fixpoint of the function $\varphi(x)$, respectively.

Under this characterization of fixpoint operators, we find that many interesting properties of the Kripke model can be represented by modal μ-formulas. For example, consider the formula $\varphi_1 = \mu x.(\Diamond x \vee p)$. For every Kripke model $\mathcal{S}$ and its possible world s, we have $\mathcal{S}, s \models \varphi_1$ if and only if there is some possible world reachable from s in which p is true. Consider the formula $\varphi_2 = \nu y.\mu x.((\Diamond y \wedge p) \vee (\Diamond x \wedge \neg p))$. Then $\mathcal{S}, s \models \varphi_2$ if and only if there is some path from s on which p is true infinitely often.

2.3 Axiomatization

We give the Kozen's axiomatization Koz for the modal μ-calculus in the Tait-style calculus.[5] Hereafter, we will write Γ, Δ, ... for a finite set of formulas. Moreover, the standard abbreviation in the Tait-style calculus are used. That is, we write α, Γ for $\{\alpha\} \cup \Gamma$; Γ, Δ for $\Gamma \cup \Delta$; and $\sim\Delta$ for $\{\sim\delta \mid \delta \in \Delta\}$ and so forth.

Axioms Koz contains basic tautologies of classical propositional calculus and the *pre-fixpoint axioms*:

$$\frac{}{\bot \vdash}\ (\mathsf{Bot}) \qquad \frac{}{\varphi, \sim\varphi \vdash}\ (\mathsf{Tau}) \qquad \frac{}{\alpha(\mu x.\alpha(x)), \sim\mu x.\alpha(x) \vdash}\ (\mathsf{Prefix})$$

Inference Rules In addition to the classical inference rules from propositional modal logic, for any formula $\varphi(x)$ such that x appears only positively, we have the *induction rule* (Ind) to handle fixpoints:

$$\frac{\alpha, \Gamma \vdash \quad \beta, \Gamma \vdash}{\alpha \vee \beta, \Gamma \vdash}\ (\vee) \qquad \frac{\alpha, \beta, \Gamma \vdash}{\alpha \wedge \beta, \Gamma \vdash}\ (\wedge)$$

$$\frac{\Gamma \vdash}{\alpha, \Gamma \vdash}\ (\mathsf{Weak}) \qquad \frac{\psi, \{\alpha \mid \Box\alpha \in \Gamma\} \vdash}{\Diamond\psi, \Gamma \vdash}\ (\Diamond)$$

$$\frac{\Gamma, \sim\alpha \vdash \quad \alpha, \Delta \vdash}{\Gamma, \Delta \vdash}\ (\mathsf{Cut}) \qquad \frac{\varphi(\psi), \sim\psi \vdash}{\mu x.\varphi(x), \sim\psi \vdash}\ (\mathsf{Ind})$$

Of course, the condition of substitution is satisfied in the (Ind)-rule; namely, no new binding relation occurs by applying the substitution $\varphi(\psi)$. As usual, we say that a formula $\sim\bigwedge\Gamma$ is *provable* in Koz (denoted $\Gamma \vdash$) if there exists a proof diagram of Γ. We frequently use notation such as $\Gamma \vdash \Delta$ to mean $\Gamma, \sim\Delta \vdash$.

The following two lemmas state some basic properties of Koz. We leave the proofs of these statement as an exercise to the reader.

Lemma 2.9. *Let φ be a modal μ-formula and let $\alpha(x)$ and $\beta(x,x)$ be modal μ-formulas where x appears only positively. Then, the following holds:*

[5] In Kozen's original article [8], the system Koz was defined as the axiomatization of the equational theory. Nevertheless we present Koz as an equivalent Tait-style calculus due to the calculus' affinity with the tableaux discussed in the sequel.

1. $\vdash \sigma x.\alpha(x) \leftrightarrow \sigma y.\alpha(y)$ *where* $y \notin \mathsf{Free}(\alpha(x))$.

2. $\vdash \sigma x.\beta(x,x) \leftrightarrow \sigma x.\sigma y.\beta(x,y)$ *where* $y \notin \mathsf{Free}(\beta(x,x))$.

3. $\vdash \mu x.\alpha(x) \leftrightarrow \alpha(\bot)$*, if no appearances of* x *are in the scope of any modal operators.*

4. $\vdash \nu x.\alpha(x) \leftrightarrow \alpha(\top)$*, if no appearances of* x *are in the scope of any modal operators.*

5. *We can construct a well-named formula* $\mathsf{wnf}(\varphi) \in \mathsf{WNF}$ *such that* $\vdash \varphi \leftrightarrow \mathsf{wnf}(\varphi)$.

Lemma 2.10. *Let* α*,* β*,* $\varphi(x)$*,* $\psi(x)$*,* $\chi_1(x)$ *and* $\chi_2(x)$ *be modal* μ*-formulas where* x *appears only positively in* $\varphi(x)$ *and* $\psi(x)$*. Further, suppose that* $\chi_1(\alpha)$*,* $\chi_1(\beta)$ *and* $\chi_2(\alpha)$ *are legal substitution; namely, a new binding relation does not occur by such substitutions. Then, the following holds:*

1. *If* $\vdash \varphi(x) \to \psi(x)$ *then* $\vdash \sigma x.\varphi(x) \to \sigma x.\psi(x)$.

2. *If* $\vdash \alpha \leftrightarrow \beta$ *then* $\vdash \chi_1(\alpha) \leftrightarrow \chi_1(\beta)$.

3. *If* $\vdash \chi_1(x) \leftrightarrow \chi_2(x)$ *then* $\vdash \chi_1(\alpha) \leftrightarrow \chi_2(\alpha)$.

Remark 2.11 (Substitution). Let $\varphi(x)$ and ψ be formulas where $\varphi(x) = \varphi(x_1, \dots, x_k)[x_1/x, \dots, x_k/x]$ and $x \notin \mathsf{Free}(\varphi(x_1, \dots, x_k))$; i.e., $\varphi(x_1, \dots, x_k)$ is a formula obtained by renaming all instances of x in $\varphi(x)$. Let $\varphi'(x)$ be the formula obtained by renaming bound variables in $\varphi(x)$ and let ψ_i with $1 \le i \le k$ be formulas obtained by renaming bound variables in ψ so that;

$$\mathsf{Bound}(\varphi'(x)) \cap \mathsf{Free}(\varphi'(x)) = \emptyset \tag{1}$$
$$\mathsf{Bound}(\varphi'(x)) \cap \mathsf{Free}(\psi_i) = \emptyset \quad (1 \le \forall i \le k) \tag{2}$$
$$\mathsf{Free}(\varphi'(x)) \cap \mathsf{Bound}(\psi_i) = \emptyset \quad (1 \le \forall i \le k) \tag{3}$$
$$\mathsf{Bound}(\psi_i) \cap \mathsf{Free}(\psi_j) = \emptyset \quad (1 \le \forall i, \forall j \le k) \tag{4}$$
$$\mathsf{Bound}(\psi_i) \cap \mathsf{Bound}(\psi_j) = \emptyset \quad (1 \le \forall i, \forall j \le k,\ i \neq j) \tag{5}$$

Then the formula $\varphi'(\psi_1, \dots, \psi_k)$ is termed well-named. Moreover, from Lemmas 2.9 and 2.10, we can assume that $\varphi'(\psi_1, \dots, \psi_k)$ is syntactically (and thus semantically) equivalent to $\varphi(\psi)$. Hereafter, we will assume that $\varphi(\psi)$ is an abbreviation for $\varphi'(\psi_1, \dots, \psi_k)$; this abbreviation is harmless as far as provability and satisfiability are concerned. Furthermore, we can write $\varphi(\psi)$ even if a new binding relation occurs by the substitution; in this case, we will regard it as merely an abbreviation for $\varphi'(\psi_1, \dots, \psi_k)$.

3 Automata

The purpose of this section is to define the *parity automata* and introduce a classical result concerning the complement of an ω-language characterized by some parity automaton, namely, the *Complementation Lemma*.

A parity automaton is a quintuple $\mathcal{A} = (Q, \Sigma, \delta, q_I, \Omega)$ where:

- Q is a finite set of *states* of the automaton,
- Σ is an *alphabet*,
- $q_I \in Q$ is a state called the *initial state*,
- $\delta : Q \times \Sigma \to \mathcal{P}(Q)$ is a *transition function*, and
- $\Omega : Q \to \omega$ is called the *priority function*.

Using the usual definitions, we say that $\mathcal{A}$ is *deterministic* if $|\delta(q,a)| = 1$ for every $q \in Q$ and $a \in \Sigma$. Let $\mathcal{A} = (Q, \Sigma, \delta, q_I, \Omega)$ be a parity automaton. A *run* of $\mathcal{A}$ on an ω-word $\pi \in \Sigma^\omega$ is an infinite sequence $\xi \in Q^\omega$ of a state where $\xi[1] = q_I$ and $\xi[n+1] \in \delta(\xi[n], \pi[n])$ for any $n \ge 1$. An ω-word $\pi \in \Sigma^\omega$ is *accepted* by $\mathcal{A}$ if there is a run ξ of $\mathcal{A}$ on π satisfying the following condition:

$$\max \mathsf{Inf}(\vec{\Omega}(\xi)) = 0 \pmod 2.$$

The ω-language of all ω-words accepted by $\mathcal{A}$ is denoted by $\mathcal{L}(\mathcal{A})$.

Let $\mathcal{A} = (Q, \Sigma, \delta, q_I, \Omega)$ be a parity automaton and $\pi \in \Sigma^*$. If $\mathcal{A}$ is deterministic, then the state of $\mathcal{A}$ by reading π is uniquely determined. We denote this state $\delta(q_I, \pi)$; in other words, $\delta(q_I, \pi)$ is defined inductively on the length of π by

$$\delta(q_I, \pi) := \begin{cases} q_I & (|\pi| = 0) \\ \delta\big(\delta(q_I, \pi[1, n]), \pi[n+1]\big) & (|\pi| = n+1). \end{cases}$$

Moreover, for any $\pi \in \Sigma^* \cup \Sigma^\omega$, we denote the run of $\mathcal{A}$ on α by $\vec{\delta}(q_I, \pi)$, that is,

$$\vec{\delta}(q_I, \pi) := q_I \delta(q_I, \pi[1,1]) \delta(q_I, \pi[1,2]) \delta(q_I, \pi[1,3]) \cdots \in Q^* \cup Q^\omega.$$

The following lemma shows that the complement of the ω-language characterized by a parity automaton is also characterized by some parity automaton. The proof of this lemma can be found in the literature, for example, see [6].

Lemma 3.1 (Complementation Lemma). *For any parity automaton $\mathcal{A} = (Q, \Sigma, \delta, q_I, \Omega)$, we can construct a deterministic parity automaton $\bar{\mathcal{A}}$ such that $\mathcal{L}(\bar{\mathcal{A}}) = \Sigma^\omega \setminus \mathcal{L}(\mathcal{A})$ with $2^{\mathcal{O}(|Q|^2 \log |Q|^2)}$ states and priorities bounded by $\mathcal{O}(|Q|^2)$.*

4 Games

It is well known that *Parity games* and *Evaluation games* are important tools in the modal μ-calculus. They will also play a crucial role in this article. This section introduces these games.

4.1 Parity games

A *parity game* $\mathcal{G}$ is defined in terms of an *arena* $\mathcal{A}$ and a *priority function* Ω. An arena is a (possibly infinite) directed graph $\mathcal{A} = \langle V_0, V_1, E \rangle$, where $V_0 \cap V_1 = \emptyset$ and the edge relation is $E \subseteq (V_0 \cup V_1) \times (V_0 \cup V_1)$. We call each element of $V := V_0 \cup V_1$ a *game position* of the arena. The priority function is $\Omega : V \to \omega$ where $\Omega(V)$ is a finite set.

A *play* in arena $\mathcal{A}$ can be finite or infinite. In the former case, the play is an E-sequence $\pi = v_1 \cdots v_n \in V^+$ such that $E(v_n) = \emptyset$. In the later case, the play is simply an infinite E-sequence. Thus, a finite or infinite play in a game can be seen as the trace of a token moved on the arena by two players, Player 0 and Player 1, in such a way that if the token is in position $v \in V_\delta$ ($\delta \in \{0, 1\}$), then Player δ has to choose a successor of v to which to move the token. A play π is *winning* for Player 0 if:

- If π is finite, then the last position $\pi[|\pi|]$ of the play is in V_1.
- If π is infinite, then $\max \mathsf{inf}(\vec{\Omega}(\pi)) = 0 \pmod 2$.

A play is winning for Player 1 if it is not winning for Player 0.

Example 4.1. Let $\mathcal{G} = \langle \langle V_0, V_1, E \rangle, \Omega \rangle$ be the parity game presented in Figure 1. We have the 0-vertices $V_0 = \{v_1, v_5\}$ (circles) and the 1-vertices $V_1 = \{v_2, v_3, v_4\}$ (squares). The edge relation E and priority function Ω may be derived from the figure, e.g., $\Omega(v_1) = 2$ and $\Omega(v_2) = 3$. A possible

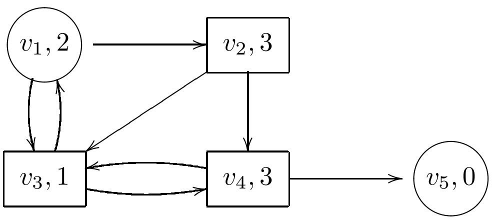

Figure 1: An example of a parity game.

infinite play in this game is, for example, $\pi = v_1 v_2 (v_3 v_1)^\omega$. This play is winning for Player 0 because $\vec{\Omega}(v_1 v_2 (v_3 v_1)^\omega) = \langle 2, 3, 1, 2, 1, 2, \ldots \rangle$ and:

$$\max \mathsf{inf}(\vec{\Omega}(\pi)) = \max \mathsf{inf}(\langle 2, 3, 1, 2, 1, 2, \ldots \rangle) = \max\{1, 2\} = 2 = 0 \pmod 2.$$

A finite E-sequence $\pi = v_1 v_2 v_4 v_5$ is also a possible play since v_5 is a dead-end. This play is winning for Player 1 because the last position v_5 is in V_0.

Let $\mathcal{A}$ be an arena. A *strategy* for Player δ with $\delta \in \{0,1\}$ is a partial function $f_\delta : V^* V_\delta \to V$ such that for any $\pi \in V^* V_\delta$, if $E(\pi[|\pi|]) \neq \emptyset$ then $f_\delta(\pi)$ is defined and satisfies $f_\delta(\pi) \in E(\pi[|\pi|])$. A play π is said to be *consistent* with f_δ if for every $n \in \omega$ such that $1 \leq n < |\pi|$, $\pi[n] \in V_\delta$ implies $f_\sigma(\pi[1,n]) = \pi[n+1]$. The strategy f_δ is said to be a *winning strategy* for Player δ if every play consistent with f_δ is winning for Player δ. A position $v \in V$ is winning for Player δ if there is a strategy f_δ such that every play consistent with f_δ which starts in v is winning for Player δ. A winning strategy f_δ is called *memoryless* if for all finite E-sequences π and π', $f_\delta(\pi) = f_\delta(\pi')$ whenever $\pi[|\pi|] = \pi'[|\pi'|]$. For parity games, we have a memoryless determinacy result.

Theorem 4.2 (Mostowski [10], Emerson and Jutla [5]). *For any parity game, one of the Players has a memoryless winning strategy from each game position.*

Considering this theorem, we will assume that all winning strategies are memoryless. In other words, a winning strategy in a parity game for Player 0 is a function $f_0 : V_0 \to V$, and is denoted analogously for Player 1.

4.2 Evaluation games

Given a well-named formula φ, a Kripke model $\mathcal{S} = (S, R, \lambda)$ and its world s_0, we define the *evaluation game* $\mathcal{EG}(\mathcal{S}, s_0, \varphi)$ as a parity game with Player 0 and 1 moving a token to positions of the form $\langle \psi, s \rangle \in \mathsf{Sub}(\varphi) \times S$. Intuitively, Player 0 asserts that "the formula φ is true at the possible world s_0" and Player 1 asserts the opposite.

The initial game position is $\langle \varphi, s_0 \rangle$. Table 1 displays the rules of the game, that is, admissible moves from a given position, and the player supposed to make this move. In order to define the priority function

Position	Player	Admissible moves
$\langle \bot, s \rangle$	0	$\emptyset$
$\langle \top, s \rangle$	1	$\emptyset$
$\langle p, s \rangle$ with $p \in \mathsf{Free}(\varphi)$ and $s \in \lambda(p)$	1	$\emptyset$
$\langle p, s \rangle$ with $p \in \mathsf{Free}(\varphi)$ and $s \notin \lambda(p)$	0	$\emptyset$
$\langle \neg p, s \rangle$ with $p \in \mathsf{Free}(\varphi)$ and $s \notin \lambda(p)$	1	$\emptyset$
$\langle \neg p, s \rangle$ with $p \in \mathsf{Free}(\varphi)$ and $s \in \lambda(p)$	0	$\emptyset$
$\langle \alpha \wedge \beta, s \rangle$	1	$\{\langle \alpha, s \rangle, \langle \beta, s \rangle\}$
$\langle \alpha \vee \beta, s \rangle$	0	$\{\langle \alpha, s \rangle, \langle \beta, s \rangle\}$
$\langle \Box \alpha, s \rangle$	1	$\{\langle \alpha, t \rangle \mid (s,t) \in R\}$
$\langle \Diamond \alpha, s \rangle$	0	$\{\langle \alpha, t \rangle \mid (s,t) \in R\}$
$\langle \sigma x.\alpha, s \rangle$	0	$\{\langle \alpha, s \rangle\}$
$\langle x, s \rangle$ with $x \in \mathsf{Bound}(\varphi)$	0	$\{\langle \varphi_x(x), s \rangle\}$

Table 1: Admissible move of $\mathcal{EG}(\mathcal{S}, s_0, \varphi)$

$\Omega_e : V \to \omega$, we define the function $\Omega_\varphi : \mathsf{Sub}(\varphi) \to \omega$ as follows:

$$\Omega_\varphi(\psi) := \begin{cases} \mathsf{alt}(\sigma_x.\varphi_x(x)) - 1 & \text{if } \psi = \varphi_x(x),\ \sigma_x = \mu \text{ and } \mathsf{alt}(\sigma_x.\varphi_x(x)) = 0 \pmod 2, \\ \mathsf{alt}(\sigma_x.\varphi_x(x)) & \text{if } \psi = \varphi_x(x),\ \sigma_x = \mu \text{ and } \mathsf{alt}(\sigma_x.\varphi_x(x)) = 1 \pmod 2, \\ \mathsf{alt}(\sigma_x.\varphi_x(x)) - 1 & \text{if } \psi = \varphi_x(x),\ \sigma_x = \nu \text{ and } \mathsf{alt}(\sigma_x.\varphi_x(x)) = 1 \pmod 2, \\ \mathsf{alt}(\sigma_x.\varphi_x(x)) & \text{if } \psi = \varphi_x(x),\ \sigma_x = \nu \text{ and } \mathsf{alt}(\sigma_x.\varphi_x(x)) = 0 \pmod 2, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Then we define $\Omega_e(\langle \psi, s \rangle) := \Omega_\varphi(\psi)$ for each game position $\langle \psi, s \rangle$.

The following theorem was proved by Streett and Emerson [12].

Theorem 4.3 (Streett and Emerson [12]). *For any well-named formula φ, Kripke model $\mathcal{S}$ and its world s, we have $\mathcal{S}, s \models \varphi$ if and only if Player 0 has a (memoryless) winning strategy for $\mathcal{EG}(\mathcal{S}, s, \varphi)$.*

5 Tableaux

In this section, we introduce the concept of a *tableau* and investigate some of its characteristic properties. The main result of this section is Corollary 5.27 in which we prove Claim (g) as foreshadowed in Section 1. This section is divided into the following three subsections.

In Subsection 5.1, we introduce the tableau and *tableau games*, which originated in Niwinski and Walukiewicz [11], with some modifications for our concept.

In Subsection 5.2, the *automaton normal form* is introduced and Claim (b) is shown; namely, for any formula φ we can construct an equivalent automaton normal form $\mathsf{anf}(\varphi)$. Although this result is not new, we will see the proof of it in detail since our argument relies on both the result and the process for proving (b).

In Subsection 5.3, we introduce the novel concept of a *wide tableau*, which is a generalization of tableaux and prove Claim (g) using this new resource.

5.1 Tableau games

Definition 5.1 (Cover modality). Let Φ be a finite set of formulas. Then $\nabla\Phi$ denotes an abbreviation of the following formula:

$$(\bigwedge \Diamond\Phi) \wedge (\Box \bigvee \Phi).$$

Here, $\Diamond\Phi$ denotes the set $\{\Diamond\varphi \mid \varphi \in \Phi\}$, and as always, we use the convention that $\bigvee\emptyset := \bot$ and $\bigwedge\emptyset := \top$. The symbol ∇ is called the *cover modality.*

Remark 5.2. Note that the both the ordinary diamond $\Diamond$ and the ordinary box $\Box$ can be expressed in term of cover modality and the disjunction:

$$\Diamond\varphi \equiv \nabla\{\varphi, \top\},$$
$$\Box\varphi \equiv \nabla\emptyset \vee \nabla\{\varphi\}.$$

Therefore, without loss of generality we restrict ourselves to using only ∇ instead of $\Diamond$ and $\Box$. Hereafter, we exclusively use cover modality notation instead of ordinal modal notation; thus *if not otherwise mentioned, all formulas are assumed to be using this new constructor*. Moreover, the concepts from Section 2 such as the well-named formula and the alternation depth extend to formulas using this modality.

Definition 5.3. Let Γ be a set of formulas. We will say that Γ is *locally consistent* if Γ does not contain $\bot$ nor any propositional variable p and its negation $\neg p$ simultaneously. On the other hand, Γ is said to be *modal* (under φ) if Γ does not contain formulas of the forms $\alpha \vee \beta$, $\alpha \wedge \beta$, $\sigma x.\alpha(x)$, or $x \in \mathsf{Bound}(\varphi)$. In other words, if Γ is modal, then Γ can possess only the elements of $\mathsf{Lit}(\varphi)$ and formulas of the form $\nabla\Phi$.

Definition 5.4 (Tableau). Let φ be a well-named formula. A set of *tableau rules* for φ is defined as follows:

$$\frac{\alpha, \Gamma \ \mid\ \beta, \Gamma}{\alpha \vee \beta, \Gamma}\ (\vee) \qquad \frac{\alpha, \beta, \Gamma}{\alpha \wedge \beta, \Gamma}\ (\wedge)$$

$$\frac{\varphi_x(x), \Gamma}{\sigma_x x.\varphi_x(x), \Gamma}\ (\sigma) \qquad \frac{\varphi_x(x), \Gamma}{x, \Gamma}\ (\mathsf{Reg})$$

$$\frac{\{\psi_k\} \cup \{\bigvee \Psi_n \mid n \in N_{\psi_k}\} \ \mid\ \text{For every } k \in \omega \text{ with } 1 \le k \le i \text{ and } \psi_k \in \Psi_k.}{\nabla\Psi_1, \dots, \nabla\Psi_i, l_1, \dots, l_j}\ (\nabla)$$

where in the (∇)-rule, $l_1, \dots, l_j \in \mathsf{Lit}(\varphi)$ and $N_{\psi_k} := \{n \in \omega \mid 1 \le n \le i,\ n \neq k\}$. Therefore, the premises of a (∇)-rule is equal to $\sum_{1 \le k \le i} |\Psi_k|$.

A *tableau* for φ is a structure $\mathcal{T}_\varphi = (T, C, r, L)$ where (T, C, r) is a tree structure and $L : T \to \mathcal{P}(\mathsf{Sub}(\varphi))$ is a *label function* satisfying the following clauses:

1. $L(r) = \{\varphi\}$.
2. Let $t \in T$. If $L(t)$ is modal and inconsistent then t has no child. Otherwise, if t is labeled by a set of formulas which fulfills the form of the conclusion of some tableau rules, then t has children which are labeled by the sets of formulas of premises of one of those tableau rules, e.g., if $L(t) = \{\alpha \vee \beta\}$, then t must have two children u and v with $L(u) = \{\alpha\}$ and $L(v) = \{\beta\}$.

3. The rule $(\triangledown)$ can be applied in t only if $L(t)$ is modal; in other words, $(\triangledown)$ is applicable when no other rule is applicable.

We call a node t a $(\triangledown)$-node if the rule $(\triangledown)$ is applied between t and its children. The notions of $(\vee)$-node, $(\wedge)$-node, (σ)-node and (Reg)-node are defined similarly.

Definition 5.5 (Modal and choice nodes). Leaves and $(\triangledown)$-nodes are called *modal nodes*. The root of the tableau and children of modal nodes are called *choice nodes*. We say that a modal node t and choice node u are *near* to each other if t is a descendant of u and between the C-sequence from u to t, there is no node in which the rule $(\triangledown)$ is applied. Similarly, we say that a modal node t' is a *next modal node* of a modal node t if t' is a descendant of t and between the C-sequence from t to t', rule $(\triangledown)$ is applied exactly once between t and its child. Note that, in some cases, a choice node may be also a modal node.

Definition 5.6 (Trace). Let φ be a well-named formula and $\mathcal{T}_\varphi = (T, C, r, L)$ be a tableau for φ. For each node $t \in T$ and its child $u \in C(t)$, we define the *trace function* $\mathsf{TR}_{tu} : L(t) \to \mathcal{P}(L(u))$ as follows:

- If t is a $(\vee)$-node where the rule applied between t and its children forms

$$\frac{\alpha, \Gamma \ \mid \ \beta, \Gamma}{\alpha \vee \beta, \Gamma}\ (\vee)$$

  then we set $\mathsf{TR}_{tu}(\gamma) := \{\gamma\}$ for every $\gamma \in \Gamma$. Further, we set $\mathsf{TR}_{tu}(\alpha \vee \beta) := \{\alpha\}$ when $L(u) = \{\alpha\} \cup \Gamma$ and set $\mathsf{TR}_{tu}(\alpha \vee \beta) := \{\beta\}$ when $L(u) = \{\beta\} \cup \Gamma$.

- If t is a $(\wedge)$-node where the rule applied between t and its child forms

$$\frac{\alpha, \beta, \Gamma}{\alpha \wedge \beta, \Gamma}\ (\wedge)$$

  then we set $\mathsf{TR}_{tu}(\gamma) := \{\gamma\}$ for every $\gamma \in \Gamma$, and set $\mathsf{TR}_{tu}(\alpha \wedge \beta) := \{\alpha, \beta\}$.

- If t is a (σ)-node where the rule applied between t and its child forms

$$\frac{\varphi_x(x), \Gamma}{\sigma_x x.\varphi_x(x), \Gamma}\ (\sigma)$$

  then we set $\mathsf{TR}_{tu}(\gamma) := \{\gamma\}$ for every $\gamma \in \Gamma$, and set $\mathsf{TR}_{tu}(\sigma_x x.\varphi_x(x)) := \{\varphi_x(x)\}$.

- If t is a (Reg)-node where the rule applied between t and its child forms

$$\frac{\varphi_x(x), \Gamma}{x, \Gamma}\ (\mathsf{Reg})$$

  then we set $\mathsf{TR}_{tu}(\gamma) := \{\gamma\}$ for every $\gamma \in \Gamma$, and set $\mathsf{TR}_{tu}(x) := \{\varphi_x(x)\}$.

- If t is a $(\triangledown)$-node where the rule applied between t and its children forms

$$\frac{\{\psi_k\} \cup \{\bigvee \Psi_n \mid n \in N_{\psi_k}\} \ \mid \ 1 \leq k \leq i,\ \psi_k \in \Psi_k.}{\triangledown\Psi_1, \ldots, \triangledown\Psi_i, l_1, \ldots, l_j}\ (\triangledown)$$

  Moreover, suppose u is labeled by $\{\psi_k\} \cup \{\bigvee \Psi_n \mid n \in N_{\psi_k}\}$ for some $k \leq i$ and $\psi_k \in \Psi_k$. Then we set $\mathsf{TR}_{tu}(\triangledown\Psi_k) := \{\psi_k\}$, $\mathsf{TR}_{tu}(\triangledown\Psi_n) := \{\bigvee \Psi_n\}$ for every $n \in N_{\psi_k}$, and $\mathsf{TR}_{tu}(l_n) := \emptyset$ for every $n \leq j$.

Take a finite or infinite C-sequence π of $\mathcal{T}_\varphi$. A *trace* tr on π is a finite or infinite sequence of $\mathsf{Sub}(\varphi)$ satisfying the following two conditions;

- $\mathsf{tr}[1] = \varphi$.
- For any $n \in \omega \setminus \{0\}$, if $\mathsf{tr}[n]$ is defined and satisfies $\mathsf{TR}_{\pi[n]\pi[n+1]}(\mathsf{tr}[n]) \neq \emptyset$, then $\mathsf{tr}[n+1]$ is also defined and satisfies $\mathsf{tr}[n+1] \in \mathsf{TR}_{\pi[n]\pi[n+1]}(\mathsf{tr}[n])$.

Note that, from the definition, for any $n \in \omega$ such that $1 \le n \le |\mathsf{tr}|$, we have $\mathsf{tr}[n] \in L(\pi[n])$. The infinite trace tr is said to be *even* if

$$\max \mathsf{Inf}(\vec{\Omega}_\varphi(\mathsf{tr})) = 0 \pmod 2.$$

Furthermore, an infinite branch π is *even* if every trace on it is even. The set of all traces on π is denoted by $\mathsf{TR}(\pi)$. $\mathsf{TR}(\pi[n,m])$ denotes the set $\{\mathsf{tr}[n,m] \mid \mathsf{tr} \in \mathsf{TR}(\pi)\}$ and may also be written $\mathsf{TR}(\pi[n], \pi[m])$. For any two factors $\mathsf{tr}[n,m]$ and $\mathsf{tr}'[n',m']$, we say $\mathsf{tr}[n,m]$ and $\mathsf{tr}'[n',m']$ are *equivalent* (denoted $\mathsf{tr}[n,m] \equiv \mathsf{tr}'[n',m']$) if, by ignoring invariant portions of the traces, they can be seen as the same sequence. For example, let;

$$\begin{array}{rlllll} \mathsf{tr}[n,n+3] = & \langle(\alpha\wedge\beta)\vee\gamma, & (\alpha\wedge\beta)\vee\gamma, & \alpha\wedge\beta, & \beta\rangle & \\ \mathsf{tr}'[n',n'+4] = & \langle(\alpha\wedge\beta)\vee\gamma, & \alpha\wedge\beta, & \alpha\wedge\beta, & \alpha\wedge\beta, & \beta\rangle \end{array}$$

then $\mathsf{tr}[n,n+3]$ and $\mathsf{tr}'[n',n'+4]$ are equivalent to each other. Let X and Y be the set of some factors of some traces. Then we write $X \Subset Y$ if for any $\mathsf{tr}[n,m] \in X$ there exists $\mathsf{tr}'[n',m'] \in Y$ such that $\mathsf{tr}[n,m] \equiv \mathsf{tr}'[n',m']$; and write $X \equiv Y$ if $X \Subset Y$ and $X \Supset Y$.

Let φ be a formula. Since $\mathcal{P}(\mathsf{Sub}(\varphi))$ is a finite set, it can be seen as an alphabet. The next lemma shows that there is an automaton $\mathcal{A}_\varphi$ which precisely detects the evenness of a branch of the tableau.

Lemma 5.7. *Let φ be a well-named formula and $\mathcal{T}_\varphi = (T, C, r, L)$ be a tableau for φ. Set $M = |\mathsf{Sub}(\varphi)|$. Then we can construct a deterministic parity automaton*

$$\mathcal{A}_\varphi = (Q, \mathcal{P}(\mathsf{Sub}(\varphi)), \delta, q_I, \Omega)$$

with $|Q| \in 2^{\mathcal{O}(M^2 \log M^2)}$ and priorities bounded by $\mathcal{O}(M^2)$ such that for any infinite branch π, $\mathcal{A}_\varphi$ accepts $\vec{L}(\pi) \in \mathcal{P}(\mathsf{Sub}(\varphi))^\omega$ if and only if π is even.

Proof. First, we construct a non-deterministic parity automaton

$$\mathcal{B}_\varphi = (Q', \mathcal{P}(\mathsf{Sub}(\varphi)), \delta', q'_I, \Omega')$$

which only accepts sequences of labels of π that are *not* even. Set $Q' := \mathsf{Sub}(\varphi) \uplus \{q'_I\}$, then $\mathcal{B}_\varphi$ has $(M+1)$ states. We define the transition function δ' so that $\delta'(q'_I, \{\varphi\}) := \{\varphi\}$ and $\delta'(\psi, \pi[n+1]) := \mathsf{TR}_{\pi[n]\pi[n+1]}(\psi)$ for any $n \ge 1$. The priority is defined as $\Omega'(q'_I) := 0$ and $\Omega'(\psi) := \Omega_\varphi(\psi) + 1$ for every $\psi \in \mathsf{Sub}(\varphi)$.

Now, $\mathcal{B}_\varphi$ is defined in such a way that a run of the automaton on $\vec{L}(\pi)$ forms one trace on π and the automaton accepts only *odd* traces. By applying the Complementation Lemma 3.1, we obtain the required automaton. ☐

Now, we define the *tableau games* introduced by Niwinski and Walukiewicz [11]. To distinguish players of this game from players of the evaluation games defined in subsection 4.2, we assume that players of a tableau game have other popular names; say Player 2 and Player 3. Intuitively, Player 2 asserts that "φ is satisfiable" and Player 3 asserts the opposite. This is justified by Lemmas 5.9 and 5.10.

Definition 5.8 (Tableau game). Let φ be a well-named formula, $\mathcal{T}_\varphi = (T, C, r, L)$ be a tableau for φ, and $\mathcal{A}_\varphi = (Q, \mathcal{P}(\mathsf{Sub}(\varphi)), \delta, q_I, \Omega)$ be an automaton given by Lemma 5.7. A *tableau game* for φ (denoted $\mathcal{TG}(\varphi)$) is a parity game played by Player 2 and Player 3 defined as follows:

Positions Let $M \subseteq T$ be the set of all modal nodes which are consistent. The positions of Player 2 are given by $V_2 := (T \setminus M)$ and the positions of Player 3 are given by $V_3 := M$; therefore the set of game positions is T. The starting position of this game is the root r.

Admissible moves In a position $t \in V_2$, Player 2 chooses the next position from $C(t)$. Note that when t is modal and locally inconsistent, Player 2 loses the game immediately since $C(t) = \emptyset$ and so she has no choice from t. In a position $t \in V_3$, Player 3 chooses the next position from $C(t)$. Note that when $L(t)$ does not contain a formula of the form $\nabla\Psi$, Player 3 loses the game immediately since $C(t) = \emptyset$ and so he has no choice from t.

Priority For any tableau node $t \in T$, we define the *automaton states* of t by $\mathsf{stat}(t) := \delta(q_I, \vec{L}(\pi))$ where π is the C-sequence starting at r and ending at t. Then, the priority of $t \in T$ is $\Omega(\mathsf{stat}(t))$.

Lemma 5.9. *Let φ be a well-named formula. If φ is satisfiable, then Player 2 has a winning strategy in the tableau game $\mathcal{TG}(\varphi)$.*

Proof. Let $\mathcal{S} = (S, R, \lambda)$ be a model and $s_0 \in S$ be a possible world such that $\mathcal{S}, s_0 \models \varphi$. From Theorem 4.3, we can assume that there exists a memoryless winning strategy f_0 for Player 0 in evaluation game $\mathcal{EG}(\mathcal{S}, s_0, \varphi)$. Now, we will construct a winning strategy for Player 2 in $\mathcal{TG}(\varphi)$ inductively; in the process of the defining the strategy, we will also define the *marking function* $\mathsf{mark} : T \to S$ simultaneously such that

> (†): If the current game position is in $t \in T$ and $\mathsf{mark}(t) = s$, then for any $\gamma \in L(t)$, Player 0 can win at the position $\langle \gamma, s \rangle$ by using the strategy f_0.

Initially, we define $\mathsf{mark}(r) := s_0$. This marking indeed satisfies (†). The remaining strategy and marking are divided into the following three cases:

- Suppose that the current position t is a $(\vee)$-node where t and its children are labeled

$$\frac{\alpha, \Gamma \;\mid\; \beta, \Gamma}{\alpha \vee \beta, \Gamma}\ (\vee)$$

  Then Player 2 must choose the next game position from these two children, say u and v which are labeled by $\{\alpha\} \cup \Gamma$ and $\{\beta\} \cup \Gamma$, respectively. By our induction assumption, we can assume that there exists a marking $\mathsf{mark}(t) = s$ which satisfies (†). Then Player 2 chooses u if and only if $f_0(\alpha \vee \beta, s) = \langle \alpha, s \rangle$. Player 2 also defines the new marking as $\mathsf{mark}(u) := \mathsf{mark}(t)$.

- Suppose the current position t is a (∇)-node where t and its children are labeled

$$\frac{\{\psi_k\} \cup \{\bigvee \Psi_n \mid n \in N_{\psi_k}\} \;\mid\; 1 \le k \le i,\ \psi_k \in \Psi_k.}{\nabla\Psi_1, \ldots, \nabla\Psi_i, l_1, \ldots, l_j}\ (\nabla)$$

  Moreover, suppose that Player 3 chooses $u \in C(t)$ which is labeled by $\{\psi_k\} \cup \{\bigvee \Psi_n \mid n \in N_{\psi_k}\}$. By our induction assumption, there is a marking $\mathsf{mark}(t) = s$ such that for any $m \in \omega$ with $1 \le m \le i$, $\langle \nabla\Psi_m, s \rangle$ is a winning position for Player 0 by using f_0. Since $\langle \nabla\Psi_k, s \rangle$ is winning for Player 0, the position $\langle \Diamond\psi_k, s \rangle$ is also winning for Player 0 (because $\nabla\Psi_k \equiv \left(\bigwedge \Diamond\Psi_k\right) \wedge \left(\Box \bigvee \Psi_k\right)$, and since $\langle \left(\bigwedge \Diamond\Psi_k\right) \wedge \left(\Box \bigvee \Psi_k\right), s \rangle$ is winning for Player 0 and Player 1 can choose the position $\langle \Diamond\psi_k, s \rangle$ from this position, $\langle \Diamond\psi_k, s \rangle$ must be winning for Player 0). Take the possible world s' such that $f_0(\Diamond\psi_k, s) = \langle \psi_k, s' \rangle$. Note that for any $n \in N_{\psi_k}$, since $\langle \nabla\Psi_n, s \rangle$ is winning for Player 0, the position $\langle \Box \bigvee \Psi_n, s \rangle$ is also winning, and thus, $\langle \bigvee \Psi_n, s' \rangle$ is winning for Player 0. Finally, Player 2 creates a new marking as $\mathsf{mark}(u) := s'$, and this marking satisfies (†) as discussed above.

- In another position t, Player 2 has at most one choice and so the strategy is determined automatically. Player 2 sets the new marking as $\mathsf{mark}(u) := \mathsf{mark}(t)$ for $u \in C(t)$.

Every marking and game position consistent with this strategy satisfies (†). In fact, it can be easily checked that our strategy satisfies the following stronger assertion;

> (‡): Let π be a finite or infinite play of $\mathcal{TG}(\varphi)$ consistent with our strategy, and let $\xi := \overrightarrow{\mathsf{mark}}(\pi) \in S^+ \cup S^\omega$ be the corresponding sequence of possible worlds. Then, for any trace tr on π,
>
> $$\langle \mathsf{tr}[1], \xi[1] \rangle \langle \mathsf{tr}[2], \xi[2] \rangle \langle \mathsf{tr}[3], \xi[3] \rangle \cdots$$
>
> is a play of $\mathcal{EG}(\mathcal{S}, s_0, \varphi)$ which is consistent with f_0.

From (‡), we can confirm that the above strategy is winning. Take an arbitrary play π of $\mathcal{TG}(\varphi)$ consistent with the strategy. Suppose π is a finite branch. In this case, for any $l \in L(\pi[|\pi|]) \cap \mathsf{Lit}(\varphi)$, by (‡), we can assume that $\mathcal{S}, s \models l$ and thus $L(\pi[|\pi|])$ must be consistent. This means that the final position $\pi[|\pi|]$ belongs to V_3 and, thus, Player 2 wins in this play. Suppose π is an infinite branch. In this case, by (‡), we can assume that every trace tr on π is even and, thus, π is also even so Player 2 wins in this play. Hence, our strategy is a winning strategy for Player 2. □

Lemma 5.10. *Let φ be a well-named formula. If Player 2 has a winning strategy in the tableau game $\mathcal{TG}(\varphi)$, then φ is satisfiable.*

Proof. Let $\mathcal{T}_\varphi = (T, C, r, L)$ be a tableau for φ, and let f_2 be a winning strategy for Player 2 in the tableau game $\mathcal{TG}(\varphi)$. Consider the tree with label $\mathcal{T}_\varphi|f_2 = (T_{f_2}, C_{f_2}, r, L_{f_2})$ which is obtained from $\mathcal{T}_\varphi$ by removing all nodes of $\mathcal{T}_\varphi$ except those used by f_2. Here, C_{f_2} and L_{f_2} are appropriate restrictions of C and L, respectively. We call the structure $\mathcal{T}_\varphi|f_2$ a *winning tree* for Player 2 derived by f_2. We also define a Kripke model $\mathcal{S} = (S, R, \lambda)$ as follows:

Possible worlds: S consists of all modal positions belonging to T_{f_2}.

Accessibility relation: For any $s, s' \in S (\subseteq T_{f_2})$, we have $(s, s') \in R$ if and only if s' is a next modal node of s.

Valuation: For any $p \in \mathsf{Prop}$ and $s \in S$, we have $s \in \lambda(p)$ if and only if $\neg p \notin L(s)$.

Note that for any $t \in T_{f_2}$, there exists exactly one modal node $s \in S$ which is near t, and so we can denote such an s by $\mathsf{mark}(t)$. From now on, we construct a winning strategy for Player 0 of the evaluation game $\mathcal{EG}(\mathcal{S}, \mathsf{mark}(r), \varphi)$. If we accomplish this task, then the Lemma follows since, from Theorem 4.3, we have $\mathcal{S}, \mathsf{mark}(r) \models \varphi$. Note that the strategy we will construct below is not necessarily memoryless.

First, Player 0 brings on a token and stores $\langle \varphi, r \rangle$ in that token. Subsequently, some element $\langle \psi, t \rangle \in \mathsf{Sub}(\varphi) \times T_{f_2}$ is stored in the token at any time. Player 0 will replace the content in the token according to the current game position of $\mathcal{EG}(\mathcal{S}, \mathsf{mark}(r), \varphi)$. It is always the case that:

(†): If $\langle \psi, t \rangle$ is in the token, then one of the following four conditions is satisfied:

(C1) Current game position is $\langle \psi, \mathsf{mark}(t) \rangle$ with $\psi \in L(t)$.

(C2) Current game position is $\langle \bigwedge \Diamond \Delta', \mathsf{mark}(t) \rangle$ with $\psi = \nabla \Delta \in L(t)$ and $\Delta' \subseteq \Delta$.

(C3) Current game position is $\langle \Box \bigvee \Delta, \mathsf{mark}(t) \rangle$ with $\psi = \nabla \Delta \in L(t)$.

(C4) Current game position is $\langle \bigvee \Delta', \mathsf{mark}(t) \rangle$ with $\nabla \Delta \in L(u)$, $\Delta' \subseteq \Delta$ and $\psi \in \Delta'$ where $\mathsf{mark}(t)$ is a next modal node of a modal node $u \in S$.

The strategy satisfying Condition (†) is straightforward. Suppose $\langle \psi, t \rangle$ is in the token and satisfies Condition (†), and (**C1**). If $\psi = \alpha \vee \beta$, then Player 0 proceeds accordingly on the C_{f_2}-path from t to a $(\vee)$-node u where $\alpha \vee \beta$ is reduced to α or β between u and $v \in C_{f_2}(u)$. Then, Player 0 chooses $\langle \alpha, \mathsf{mark}(v) \rangle (= \langle \alpha, \mathsf{mark}(t) \rangle)$ as the next position if and only if $\alpha \vee \beta$ is reduced to α between u and v and, further, replaces the content in the token by $\langle \alpha, v \rangle$ or $\langle \beta, v \rangle$ according to her choice of position. If $\psi = \alpha \wedge \beta$, then Player 1 chooses the next position from $\langle \alpha, \mathsf{mark}(t) \rangle$ or $\langle \beta, \mathsf{mark}(t) \rangle$. Player 0 proceeds according on C_{f_2}-path from t to a $(\wedge)$-node u where $\alpha \wedge \beta$ is reduced to α and β between u and $v \in C_{f_2}(u)$. Then Player 0 replaces the content in the token to $\langle \alpha, v \rangle$ or $\langle \beta, v \rangle$ according to Player 1's choice of position. The case of $\psi = x \in \mathsf{Bound}(\varphi)$ and $\psi = \sigma x.\varphi_x(x)$, Player 0 replaces the content of the token similarly to the above cases. If $\psi = \nabla \Delta$, then Player 1 chooses the next position from $\langle \bigwedge \Diamond \Delta, \mathsf{mark}(t) \rangle$ or $\langle \Box \bigvee \Delta, \mathsf{mark}(t) \rangle$. In both cases, Player 0 replaces the context in the token to $\langle \nabla \Delta, \mathsf{mark}(t) \rangle$. Therefore either Condition (**C2**) or (**C3**) is satisfied.

Suppose (**C2**) is satisfied. Then, Player 0 does nothing until the position reaches the forms $\langle \Diamond \delta, \mathsf{mark}(t) \rangle$ with $\delta \in \Delta$. In the position $\langle \Diamond \delta, \mathsf{mark}(t) \rangle$, Player 0 seeks the node $u \in C_{f_2}(\mathsf{mark}(t)) (= C(\mathsf{mark}(t)))$ in which $\nabla \Delta$ is reduced to δ. Then Player 0 chooses the position $\langle \delta, \mathsf{mark}(u) \rangle$ and replaces the content of the token to $\langle \delta, u \rangle$. This game position and the content in the token satisfy (**C1**).

Suppose (**C3**) is satisfied. In this case, Player 1 chooses the next position $\langle \bigvee \Delta, \mathsf{mark}(u) \rangle$ with $u \in C_{f_2}(\mathsf{mark}(t))$. If $\nabla \Delta \in L(t)$ is reduced to $\bigvee \Delta$ in u, then Player 0 replaces the content in the token to $\langle \bigvee \Delta, u \rangle$; therefore, (**C1**) is satisfied in this case. If $\nabla \Delta \in L(t)$ is reduced to $\delta \in \Delta$ in u, then Player 0 replaces the content in the token to $\langle \delta, u \rangle$; therefore, (**C4**) is satisfied in this case.

Suppose (**C4**) is satisfied. In this case, from the current game position $\langle \bigvee \Delta', \mathsf{mark}(t) \rangle$ Player 0 chooses the next position $\langle \bigvee \Delta'', \mathsf{mark}(t) \rangle$ such that $\psi \in \Delta''$. By repeating this choice, Player 0 can reach the position $\langle \psi, \mathsf{mark}(t) \rangle$. Then, the content in the token and the current game position satisfy (**C1**).

Let ξ be a play of $\mathcal{EG}(\mathcal{S}, \mathsf{mark}(r), \varphi)$ consistent with our strategy. If ξ is finite, then for $\xi[|\xi|] = \langle l, \mathsf{mark}(t) \rangle$, we have $l \in L(\mathsf{mark}(t))$ and, thus, from the definition of λ, we have $\mathcal{S}, \mathsf{mark}(t) \models l$. This means ξ is winning for player 0. Let ξ be infinite. Then, from the construction of the strategy, we can find the branch π of $\mathcal{T}|f_2$ and the trace tr on π such that

$$\xi = \langle \mathsf{tr}[1], \mathsf{mark}(\pi[1]) \rangle \langle \mathsf{tr}[2], \mathsf{mark}(\pi[2]) \rangle \langle \mathsf{tr}[3], \mathsf{mark}(\pi[3]) \rangle \dots \qquad (7)$$

Since π is a play of the tableau game $\mathcal{TG}(\varphi)$ consistent with f_2, π is even, and so tr is also even. From (7) we know that ξ is even and, thus, winning for Player 0. From the above argument, ξ is winning for Player 0 in either case and, thus, our strategy is winning for Player 0. $\square$

5.2 Automaton normal form

Definition 5.11 (Indexed tops). For technical reasons, we now expand our language by adding *indexed tops* $\mathsf{Top} := \{\top_i \mid i \in I\}$ where I is an infinite countable set of indices. Each $\top_i$ is treated like $\top$, e.g., $\top_i$ belongs to the literal, $\sim\top_i := \bot$, and for any model $\mathcal{S}$ and its world s, we have $\mathcal{S}, s \models \top_i$.

Definition 5.12 (Automaton normal form). The set of an *automaton normal form* ANF is the smallest set of formulas defined by the following clauses:

1. If $l_1, \dots, l_i \in \mathsf{Lit}$, then $\bigwedge_{1 \le j \le i} l_j \in \mathsf{ANF}$.
2. If $\alpha \vee \beta \in \mathsf{ANF}$, $\mathsf{Bound}(\alpha) \cap \mathsf{Free}(\beta) = \emptyset$ and $\mathsf{Free}(\alpha) \cap \mathsf{Bound}(\beta) = \emptyset$, then $\alpha \vee \beta \in \mathsf{ANF}$.
3. If $\alpha(x) \in \mathsf{ANF}$ where x occurs only positively in the scope of some modal operator (cover modality), and $\mathsf{Sub}(\alpha(x))$ does not contain a formula of the form $x \wedge \beta$. Then, $\sigma\vec{x}.\alpha(\vec{x}) \in \mathsf{ANF}$ where $\sigma\vec{x}.\alpha(\vec{x})$ is the abbreviation of $\sigma x_1. \dots \sigma x_k.\alpha(x_1, \dots, x_k)$ as stated in Definition 2.3.
4. If $\Phi \subseteq \mathsf{ANF}$ is a finite set such that for any $\varphi_1, \varphi_2 \in \Phi$, we have $\mathsf{Bound}(\varphi_1) \cap \mathsf{Free}(\varphi_2) = \emptyset$, then $(\nabla\Phi) \wedge (\bigwedge_{1 \le i \le j} l_i) \in \mathsf{ANF}$ where $l_1, \dots, l_j \in \mathsf{Lit} \setminus \bigcup_{\varphi \in \Phi} \mathsf{Bound}(\varphi)$ with $0 \le j$.
5. If $\alpha \in \mathsf{ANF}$ then $\alpha \wedge \top_i \in \mathsf{ANF}$.

Note that the above clauses imply $\mathsf{ANF} \subseteq \mathsf{WNF}$.

Remark 5.13. For any automaton normal form $\widehat{\varphi}$, a tableau $\mathcal{T}_{\widehat{\varphi}} = (T, C, r, L)$ for $\widehat{\varphi}$ forms very simple shapes. Indeed, for any node $t \in T$, there exists at most one formula $\widehat{\alpha} \in L(t)$ which includes some bound variables. Note that for any infinite trace tr, $\mathsf{tr}[n]$ must include some bound variables. Consequently, for any infinite branch of the tableau for an automaton normal form, there exists a unique trace on it.

Definition 5.14 (Tableau bisimulation). Let $\mathcal{T}_\alpha = (T, C, r, L)$ and $\mathcal{T}_\beta = (T', C', r', L')$ be two tableaux for some well-named formulas α and β. Let T_m and T'_m be sets of modal nodes of $\mathcal{T}_\alpha$ and $\mathcal{T}_\beta$, respectively, and let T_c and T'_c be a set of choice nodes of $\mathcal{T}_\alpha$ and $\mathcal{T}_\beta$, respectively. Then $\mathcal{T}_\alpha$ and $\mathcal{T}_\beta$ are said to be *tableau bisimilar* (notation: $\mathcal{T}_\alpha \rightleftharpoons \mathcal{T}_\beta$) if there exists a binary relation $Z \subseteq (T_m \times T'_m) \cup (T_c \times T'_c)$ satisfying the following seven conditions:

Root condition: $(r, r') \in Z$.

Prop condition: For any $t \in T_m$ and $t' \in T'_m$, if $(t, t') \in Z$, then

$$(L(t) \cap \mathsf{Lit}(\alpha)) \setminus \mathsf{Top} = (L'(t') \cap \mathsf{Lit}(\beta)) \setminus \mathsf{Top}.$$

Consequently $L(t)$ is consistent if and only if $L'(t')$ is consistent.

Forth condition on modal nodes: Take $t \in T_m$, $u \in T_c$ and $t' \in T'_m$ arbitrarily. If $(t, t') \in Z$ and $u \in C(t)$, then there exists $u' \in C'(t')$ such that $(u, u') \in Z$ (See Figure 2).

Back condition on modal nodes: The converse of the forth condition on modal nodes: Take $t \in T_m$, $t' \in T'_m$ and $u' \in T'_c$ arbitrarily. If $(t, t') \in Z$ and $u' \in C'(t')$, then there exists $u \in C(t)$ such that $(u, u') \in Z$.

Forth condition on choice nodes: Take $u \in T_c$, $t \in T_m$ and $u' \in T'_c$ arbitrarily. If $(u, u') \in Z$ and t is near u, then there exists $t' \in T'_m$ such that $(t, t') \in Z$ and t' is near u' (See Figure 2).

Back condition on choice nodes: The converse of the forth condition on choice nodes: Take $u \in T_c$, $u' \in T'_c$ and $t' \in T'_m$ arbitrarily. If $(u, u') \in Z$ and t' is near u', then there exists $t \in T_m$ such that $(t, t') \in Z$ and t is near u.

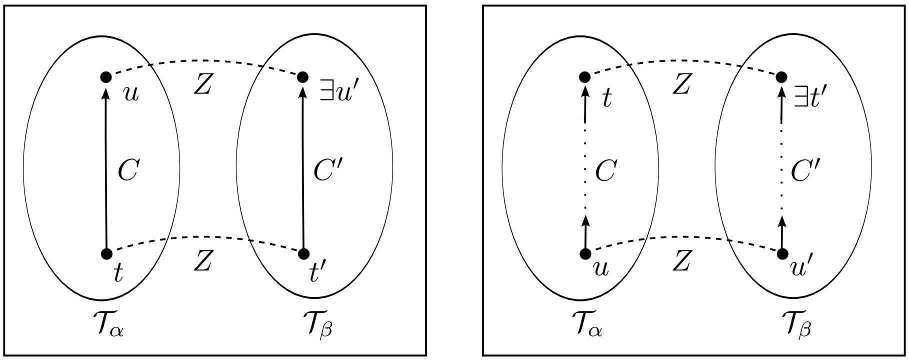

Figure 2: The forth conditions.

Parity condition: Let π and π' be infinite branches of $\mathcal{T}_\alpha$ and $\mathcal{T}_\beta$, respectively. We say that π and π' are *associated* with each other if the k-th modal nodes $\pi[i_k]$ and $\pi'[i'_k]$ satisfy $(\pi[j_k], \pi'[j'_k]) \in Z$ for any $k \in \omega \setminus \{0\}$. For any π and π' which are associated with each other, we have π is even if and only if π' is even.

If $\mathcal{T}_\alpha$ and $\mathcal{T}_\beta$ are tableau bisimilar with Z, then Z is called a *tableau bisimulation* from $\mathcal{T}_\alpha$ to $\mathcal{T}_\beta$.

Remark 5.15. As will be shown in Lemma 5.16, if $\mathcal{T}_\alpha$ and $\mathcal{T}_\beta$ are tableau bisimilar, then, α and β are semantically equivalent. However, the reverse is not applied. For example, consider the following two tableaux, say $\mathcal{T}_1$ and $\mathcal{T}_2$:

$$\dfrac{\dfrac{\dfrac{p,q \;\mid\; p,r}{p, q\vee r}(\vee)}{p\wedge(q\vee r), q\vee r}(\wedge)}{(p\wedge(q\vee r))\wedge(q\vee r)}(\wedge) \qquad \dfrac{\dfrac{\dfrac{\dfrac{p,q \;\mid\; p,q,r}{p,q\vee r,q}(\vee)}{p\wedge(q\vee r),q}(\wedge) \;\mid\; \dfrac{\dfrac{p,q,r \;\mid\; p,r}{p,q\vee r,r}(\vee)}{p\wedge(q\vee r),r}(\wedge)}{p\wedge(q\vee r),q\vee r}(\vee)}{(p\wedge(q\vee r))\wedge(q\vee r)}(\wedge)$$

In this example, even $\mathcal{T}_1$ and $\mathcal{T}_2$ are tableaux for the same formula $(p \wedge (q \vee r)) \wedge (q \vee r)$, there does not exist a tableau bisimulation between them. Because, $\mathcal{T}_2$ has leaves labeled by $\{p, q, r\}$ but $\mathcal{T}_1$ does not. Note that if $\widehat{\varphi}$ is an automaton normal form, then the tableau $\mathcal{T}_{\widehat{\varphi}}$ for $\widehat{\varphi}$ is uniquely determined.

Lemma 5.16. *Let α, β be well-named formulas. If $\mathcal{T}_\alpha \rightleftharpoons \mathcal{T}_\beta$, then $\models \alpha \leftrightarrow \beta$.*

Proof. First, we will introduce the notion of a *marking relation*, which is a slight generalization of the marking function discussed in the proof of Lemmas 5.9 and 5.10. Let $\mathcal{T}_\varphi = (T, C, r, L)$ be a tableau for some well-named formula φ, and $\mathcal{S} = (S, R, \lambda)$ be a model and $s_0 \in S$ be its possible world. The marking relation $\mathsf{Mark} \subseteq T \times S$ between $\mathcal{T}_\varphi$ and $\langle \mathcal{S}, s_0 \rangle$ is a relation satisfying the following clauses;

- $(r, s_0) \in \mathsf{Mark}$
- If $(t, s) \in \mathsf{Mark}$ and t is a choice node, then there exists modal node $u \in C^*(t)$ such that u is near t and $(u, s) \in \mathsf{Mark}$.
- If $(t, s) \in \mathsf{Mark}$ and t is a modal node, then for any $u \in C(t)$, there exists $s' \in R(s)$ such that $(u, s') \in \mathsf{Mark}$.
- If $(t, s) \in \mathsf{Mark}$, t is a modal node and $C(t) \neq \emptyset$, then for any $s' \in R(s)$, there exists $u \in C(t)$ such that $(u, s') \in \mathsf{Mark}$.
- For any modal node $t \in T$ and possible world $s \in S$ such that $(t, s) \in \mathsf{Mark}$, if $l \in L(t) \cap \mathsf{Lit}(\varphi)$, then $\mathcal{S}, s \models l$.
- For any infinite branch π such that $\{n \in \omega \mid \exists s \in S; (\pi[n], s) \in \mathsf{Mark}\}$ is infinite, π is even.

Then the following assertion holds:

(†): $\mathcal{S}, s_0 \models \varphi$ if and only if there exists a marking relation between $\mathcal{T}_\varphi$ and $\langle \mathcal{S}, s_0 \rangle$.

(†) is provable in the same method as the proofs of Lemmas 5.9 and 5.10. We leave the proof of (†) as an exercise to the reader.

Suppose $\mathcal{T}_\alpha \rightleftharpoons \mathcal{T}_\beta$ and so there exists a bisimulation Z from $\mathcal{T}_\alpha$ to $\mathcal{T}_\beta$. Then, the converse relation $Z^- := \{(t', t) \mid (t, t') \in Z\}$ is a bisimulation from $\mathcal{T}_\beta$ to $\mathcal{T}_\alpha$, and thus $\mathcal{T}_\beta \rightleftharpoons \mathcal{T}_\alpha$. Therefore, it is enough to show that $\models \alpha \to \beta$. Take a model $\mathcal{S} = (S, R, \lambda)$ and its world s_0 such that $\mathcal{S}, s_0 \models \alpha$. Then by (†), there exists a marking relation Mark' between $\mathcal{T}_\alpha$ and $\langle \mathcal{S}, s_0 \rangle$. Consider the composition

$$\mathsf{Mark} := Z^- \mathsf{Mark}' = \{(t, s) \mid (t, t') \in Z^-,\ (t', s) \in \mathsf{Mark}'\}.$$

Then Mark is a marking relation between $\mathcal{T}_\beta$ and $\langle \mathcal{S}, s_0 \rangle$; thus, from (†), we have $\mathcal{S}, s_0 \models \beta$. Therefore, we obtain $\models \alpha \to \beta$. □

Theorem 5.17 (Janin and Walukiewicz [7]). *For any well-named formula α, we can construct an automaton normal form $\mathsf{anf}(\alpha)$ such that $\mathcal{T}_\alpha \rightleftharpoons \mathcal{T}_{\mathsf{anf}(\alpha)}$ for some tableau $\mathcal{T}_\alpha$ for α.*

Proof. Let $\mathcal{T}'_\alpha = (T, C, r, L)$ be a tableau for a given formula α, $\mathcal{A}_\alpha = (Q, \mathcal{P}(\mathsf{Sub}(\alpha)), \delta, q_I, \Omega)$ be an automaton as given by Lemma 5.7, and $\mathsf{stat}(t)$ be the automaton states of $t \in T$ as defined in Definition 5.8.

First, we construct a tableau-like structure $\mathcal{TB}_\alpha = (T_b, C_b, r_b, L_b, B_b)$ called a *tableau with back edge* from $\mathcal{T}'_\alpha$ as follows:

- The node $t \in T$ is called a *loop node* if;

  (♠) There is a proper ancestor u such that $\langle L(t), \mathsf{stat}(t)\rangle = \langle L(u), \mathsf{stat}(u)\rangle$, and

  (♡) for any $v \in T$ such that $v \in C^*(u)$ and $t \in C^*(v)$, we have $\Omega(\mathsf{stat}(v)) \leq \Omega(\mathsf{stat}(t))(= \Omega(\mathsf{stat}(u)))$.

  In this situation, the node u is called a *return node* of t. Note that for any infinite branch π of $\mathcal{T}_\alpha$, there exists a loop node on π. Indeed, take $N := \max \Omega(\mathsf{Inf}(\vec{\mathsf{stat}}(\pi)))$. Then, since $\mathcal{P}(\mathsf{Sub}(\alpha)) \times Q$ is finite, there exists $\langle \Gamma, q\rangle \in \mathcal{P}(\mathsf{Sub}(\alpha)) \times Q$ such that $\Omega(q) = N$ and

  $$\mathcal{N} := \{n \in \omega \mid \langle \Gamma, q\rangle = \langle L(\pi[n]), \mathsf{stat}(\pi[n])\rangle\}$$

  is an infinite set. Take a natural number K such that for any $n > K$, we have $\Omega(\mathsf{stat}(\pi[n])) \leq N$. Moreover, take $n_1, n_2 \in \mathcal{N}$ such that $K < n_1 < n_2$. Then, from the definitions of $\mathcal{N}$ and K, we have $\langle L(\pi[n_1]), \mathsf{stat}(\pi[n_1])\rangle = \langle L(\pi[n_2]), \mathsf{stat}(\pi[n_2])\rangle$ and for any $k \in \omega$ such that $n_1 \leq k \leq n_2$, $\Omega(\mathsf{stat}(\pi[k])) \leq \Omega(\mathsf{stat}(\pi[n_2]))$. Therefore $\pi[n_2]$ is a loop node with return node $\pi[n_1]$.

  We define the set T_b of nodes as follows:

  $$T_b := \{t \in T \mid \text{for any proper ancestor } u \text{ of } t,\ u \text{ is } \textit{not} \text{ a loop node}\}$$

  Intuitively speaking, we trace the nodes on each branch from the root and as soon as we arrive at a return node, we cut off the former branch from the tableau.

- Set $C_b := C|_{T_b \times T_b}$, $r_b := r$ and $L_b := L|_{T_b}$.

- $B_b := \{(t, u) \in T_b \times T_b \mid t \text{ is a loop node and } u \text{ is a return node of } t\}$. An element of B_b is called *back edge.*

By König's lemma, we can assume that $\mathcal{TB}_\alpha$ is a finite structure because it has no infinite branches. The tableau with back edge is very similar to the basic tableau. In fact, the unwinding $\mathsf{UNW}_{r_b}(\mathcal{TB}_\alpha)$ is a tableau for α. Therefore, we use the terminology and concepts of the tableau, such as the concept of the parity of the sequence of nodes. From the definition of loop and return nodes (particularly Condition (♡)), we can assume that

> (†): Let π be an infinite $(C_b \cup B_b)$-sequence and let $t \in T_b$ be the return node which appears infinitely often in π and is nearest to the root of all such return nodes. Then, π is even if and only if $\Omega(\mathsf{stat}(t))$ is even.

Next, we assign an automaton normal form $\mathsf{anf}(t)$ to each node $t \in T_b$ by using top-down fashion:

Base step: Let $t \in T_b$ be a leaf. If t is not a loop node, then t must be a modal node with an inconsistent label or contain no formula of the form $\nabla\Phi$. In both cases, we assign $\mathsf{anf}(t) := \bigwedge_{1\le k\le i} l_k$ where $\{l_1,\dots,l_i\} = L_b(t) \cap \mathsf{Lit}(\alpha)$. If t is a loop node, we take $x_t \in \mathsf{Prop} \setminus \mathsf{Sub}(\varphi)$ uniquely for each such leaf and we set $\mathsf{anf}(t) := x_t$.

Inductive step I: Suppose $t \in T_b$ is a (∇)-node where t is labeled by $\{\nabla\Psi_1,\dots,\nabla\Psi_i, l_1,\dots,l_j\}$ with $l_1,\dots,l_j \in \mathsf{Lit}(\alpha)$, and we have already assigned the automaton normal form $\mathsf{anf}(u)$ for each child $u \in C_b(t)$. In this situation, we first assign $\mathsf{anf}^-(t)$ to t as follows:

$$\begin{aligned}\mathsf{anf}^-(t) &:= \nabla\{\mathsf{anf}(u) \mid u \in C_b(t)\} \wedge \left(\bigwedge_{1\le k\le j} l_k\right)\\ &= \left(\bigwedge_{u\in C_b(t)} \Diamond \mathsf{anf}(u)\right) \wedge \Box\left(\bigvee_{1\le k\le i}\left(\bigvee_{u\in C_b^{(k)}(t)} \mathsf{anf}(u)\right)\right) \wedge \left(\bigwedge_{1\le k\le j} l_k\right) \qquad (8)\end{aligned}$$

where $C_b^{(k)}(t)$ denotes the set of all children $u \in C_b(t)$ such that $\nabla\Psi_k$ is reduced to some $\psi_k \in \Psi_k$ between t and u. That is, we designate the order of disjunction in $\mathsf{anf}^-(t)$ for technical reasons (see Remark 5.18). If t is not a return node, then we set $\mathsf{anf}(t) := \mathsf{anf}^-(t)$. Alternatively, if t is a return node, then let $t_1,\dots,t_n$ be all the loop nodes such that $(t_k,t) \in B_b$ $(1 \le k \le n)$. We set

$$\sigma_t := \begin{cases} \mu & \text{If } \Omega(\mathsf{stat}(t))(=\Omega(\mathsf{stat}(t_1)) = \cdots = \Omega(\mathsf{stat}(t_n))) = 1 \pmod 2\\ \nu & \text{If } \Omega(\mathsf{stat}(t))(=\Omega(\mathsf{stat}(t_1)) = \cdots = \Omega(\mathsf{stat}(t_n))) = 0 \pmod 2\end{cases} \qquad (9)$$

In this case we define $\mathsf{anf}(t)$ as $\mathsf{anf}(t) := \sigma_t x_{t_1}.\dots.\sigma_t x_{t_n}.\mathsf{anf}^-(t)$.

Inductive step II: Suppose $t \in T_b$ is a $(\vee)$-node where, for both children $u, v \in C_b(t)$, we have already assigned the automaton normal forms $\mathsf{anf}(u)$ and $\mathsf{anf}(v)$, respectively. If t is not a return node, then we set $\mathsf{anf}(t) := \mathsf{anf}(u) \vee \mathsf{anf}(v)$. Suppose t is a return node. Let $t_1,\dots,t_n$ be all the loop nodes such that $(t_k,t) \in B_b$ $(1 \le k \le n)$. In this case, σ_t is defined in the same way as (9) and we define $\mathsf{anf}(t)$ as $\mathsf{anf}(t) := \sigma_t x_{t_1}.\dots.\sigma_t x_{t_n}.\big(\mathsf{anf}(u) \vee \mathsf{anf}(v)\big)$.

Inductive step III: Suppose $t \in T_b$ is a $(\wedge)$-, (σ)- or (Reg)-node where we have already assigned the automaton normal form $\mathsf{anf}(u)$ for the child $u \in C_b(t)$. If t is not a return node, then we assign $\mathsf{anf}(t) := \mathsf{anf}(u) \wedge \top_t$ where $\top_t$ is an indexed top which is taken uniquely for each $t \in T_b$. If t is a return node and $t_1,\dots,t_n$ are all the loop nodes such that $(t_k,t) \in B_b$ $(1 \le k \le n)$, then, σ_t is defined in the same way as (9), and we define $\mathsf{anf}(t)$ as $\mathsf{anf}(t) := \sigma_t x_{t_1}.\dots.\sigma_t x_{t_n}.\,\mathsf{anf}(u)$.

We take $\mathsf{anf}(\alpha) := \mathsf{anf}(r_b)$.

Consider the structure $(T_b, C_b, r_b, \mathsf{anf}, B_b)$. We intuit that this structure is almost a tableau with back edge for $\mathsf{anf}(\alpha)$. To clarify this intuition, we give a structure $\mathcal{TB}_{\mathsf{anf}(\alpha)} = (\widehat{T}, \widehat{C}, \widehat{r}, \widehat{L}, \widehat{B})$ by applying the following four steps of procedure re-formatting $(T_b, C_b, r_b, \mathsf{anf}, B_b)$ so that $\mathcal{TB}_{\mathsf{anf}(\alpha)}$ can be seen as a proper tableau with back edge. At the same time, we define the relation $Z^+ \subseteq T_b \times \widehat{T}$.

Step I (insert (σ)-nodes) Initially, we set $(\widehat{T}, \widehat{C}, \widehat{r}, \widehat{L}, \widehat{B}) := (T_b, C_b, r_b, \widehat{L}, B_b)$ where $\widehat{L}(t) := \{\mathsf{anf}(t)\}$, and set $Z^+ := \{(t,t) \mid t \in T_b\}$. Let $t \in \widehat{T}$ be a return node where $t_1,\dots,t_n$ are all the loop nodes such that $(t_k,t) \in \widehat{B}$ $(1 \le k \le n)$. Then, we insert the (σ)-nodes $u_1,\dots,u_n$ between t and its children in such a way that

$$\mathsf{anf}(t) = \sigma_t x_{t_1}.\sigma_t x_{t_2}.\dots.\sigma_t x_{t_n}.\beta(x_{t_1},\dots,x_{t_n})$$

is reduced to $\beta(x_{t_1},\dots,x_{t_n})$ from u_1 to u_n.[6] Moreover, we expand the relation Z^+ by adding $\{(t,u_k) \mid 1 \le k \le n\}$. For example, if t is a $(\vee)$-node in $\mathcal{TB}_\alpha$ such that $\{v_1,v_2\} = C_b(t)$, then our

[6] In other words, we add $u_1,\dots,u_n$ into $\widehat{T}$, add $(t,u_1),(u_1,u_2),\dots,(u_{n-1},u_n)$ and $\{(u_n,u) \mid u \in \widehat{C}(t)\}$ into $\widehat{C}$, discard $\{(t,u) \mid u \in \widehat{C}(t)\}$ from $\widehat{C}$, and expand $\widehat{L}$ to $u_1,\dots,u_n$ appropriately.

procedure would be as follows:

$$\frac{\mathsf{anf}(v_1) \;\mid\; \mathsf{anf}(v_2)}{\sigma_t x_{t_1}.\sigma_t x_{t_2}.\dots.\sigma_t x_{t_n}.\,(\mathsf{anf}(v_1)\vee\mathsf{anf}(v_2))} \quad\Rightarrow\quad \cfrac{\cfrac{\cfrac{\mathsf{anf}(v_1) \;\mid\; \mathsf{anf}(v_2)}{\mathsf{anf}(v_1)\vee\mathsf{anf}(v_2)}\,(\vee)}{\vdots\,(\sigma)}}{\cfrac{\sigma_t x_{t_2}.\dots.\sigma_t x_{t_n}.\,(\mathsf{anf}(v_1)\vee\mathsf{anf}(v_2))}{\sigma_t x_{t_1}.\sigma_t x_{t_2}.\dots.\sigma_t x_{t_n}.\,(\mathsf{anf}(v_1)\vee\mathsf{anf}(v_2))}\,(\sigma)}$$

Step II (insert $(\wedge)$**-nodes)** Let $t \in \widehat{T}$ be a node which is labeled by;

$$\nabla\{\mathsf{anf}(u) \mid u \in \widehat{C}(t)\} \wedge \left(\bigwedge_{1\le k\le j} l_k\right).$$

Then, we insert the $(\wedge)$-nodes $u_0, \dots, u_i$ between t' and its children (i.e., the nodes of $\widehat{C}(t)$) and label such $u_1, \dots, u_j$ as below:

$$\frac{\mathsf{anf}(u) \;\mid\; u \in \widehat{C}(t)}{\nabla\{\mathsf{anf}(u) \mid u \in \widehat{C}(t)\} \wedge \left(\bigwedge_{1\le k\le j} l_k\right)} \quad\Rightarrow\quad \cfrac{\cfrac{\cfrac{\mathsf{anf}(u) \;\mid\; u \in \widehat{C}(t)}{\nabla\{\mathsf{anf}(u) \mid u \in \widehat{C}(t)\}, l_1, \dots, l_j}\,(\nabla)}{\vdots\,(\wedge)}}{\cfrac{\nabla\{\mathsf{anf}(u) \mid u \in \widehat{C}(t)\}, \left(\bigwedge_{1\le k\le j} l_k\right)}{\nabla\{\mathsf{anf}(u) \mid u \in \widehat{C}(t)\} \wedge \left(\bigwedge_{1\le k\le j} l_k\right)}\,(\wedge)}$$

Further, we expand the relation Z^+ by adding $\{(t, u_k) \mid 1 \le k \le j\}$.

Step III (revise the back edges) Let t_k with $1 \le k \le n$ be the loop node, and t be the return node of t_k such that

$$\mathsf{anf}(t_k) = x_{t_k}$$
$$\mathsf{anf}(t) = \sigma_t x_{t_1}.\sigma_t x_{t_2}.\dots.\sigma_t x_{t_n}.\beta(x_{t_1}, \dots, x_{t_n}).$$

If $2 \le k$, then we delete (t_k, t) from $\widehat{B}$ and add (t_k, u_k) into $\widehat{B}$ where u_k is the unique nodes satisfying;

$$\widehat{L}(u_k) = \{\sigma_t x_{t_k}.\dots.\sigma_t x_{t_n}.\beta(x_{t_1}, \dots, x_{t_n})\}.$$

By this revising procedure, for any loop node t and its return node u, $\widehat{L}(t)$ and $\widehat{L}(u)$ form the (Reg)-rule of $\mathsf{anf}(\alpha)$.

Step IV (add the indexed tops) Suppose $t \in \widehat{T}$ and its child u are labeled as follows;

$$\frac{\mathsf{anf}(u)}{\mathsf{anf}(u) \wedge \top_t}$$

Then, we add $\top_t$ to $\widehat{L}(v)$ where $v \in (\widehat{C} \cup \widehat{B})^+(t)$ such that, between the $(\widehat{C} \cup \widehat{B})$-path from t to v, there does not exist a (∇)-node. By this adding procedure, such a t becomes a proper $(\wedge)$-node.

The structure $\mathcal{TB}_{\mathsf{anf}(\alpha)} = (\widehat{T}, \widehat{C}, \widehat{r}, \widehat{L}, \widehat{B})$ repaired by the above four procedures can be seen as a tableau with back edge for $\mathsf{anf}(\alpha)$ in the sense that the following two assertions hold:

(♣) The unwinding $\mathsf{UNW}_{\widehat{r}}(\mathcal{TB}_{\mathsf{anf}(\alpha)})$ is a tableau of $\mathsf{anf}(\alpha)$.

(♢) Let $\widehat{\pi}$ be an infinite $(\widehat{C} \cup \widehat{B})$-sequence and let $\widehat{t} \in \widehat{T}$ be the return node which appears infinitely often in $\widehat{\pi}$ and is nearest to the root of all such return nodes. Then $\widehat{\pi}$ is even if and only if $\widehat{L}(\widehat{t})$ includes a ν-formula.

Set $Z := Z^{+}|_{((T_b)_m \times \widehat{T}_m) \cup ((T_b)_c \times \widehat{T}_c)}$. If we extend the relation Z to the pair of nodes of $\mathsf{UNW}_r(\mathcal{TB}_\alpha)$ and $\mathsf{UNW}_{\widehat{r}}(\mathcal{TB}_{\mathsf{anf}(\alpha)})$, then Z clearly satisfies the root condition, prop condition, back conditions and forth conditions. Moreover, from (†) and (◊), we can assume that Z satisfies the Parity condition. Therefore, we have $\mathsf{UNW}_r(\mathcal{TB}_\alpha) \rightleftharpoons \mathsf{UNW}_{\widehat{r}}(\mathcal{TB}_{\mathsf{anf}(\alpha)})$, and so $\mathcal{T}_\alpha := \mathsf{UNW}_r(\mathcal{TB}_\alpha)$ and $\mathsf{anf}(\alpha)$ satisfy the required condition. ☐

Remark 5.18. Let $\mathsf{Sub}'(\mathsf{anf}(\alpha))$ be the set of subformulas of $\mathsf{anf}(\alpha)$ which contains some bound variables. From the relation Z^{+} constructed in the proof of Lemma 5.17, we can construct a function f from $\mathsf{Sub}'(\mathsf{anf}(\alpha))$ to $\mathcal{P}(\mathsf{Sub}(\alpha))$ naturally because of the following:

- for any $\widehat{\beta} \in \mathsf{Sub}'(\mathsf{anf}(\alpha))$, there exists a unique $\widehat{t} \in \widehat{T}$ such that $\widehat{\beta} \in \widehat{L}(\widehat{t})$; and
- for any $\widehat{t} \in \widehat{T}$ there exists a unique $t \in T_b$ such that $(t, \widehat{t}) \in Z^{+}$.

Therefore, if we define $f(\widehat{\beta}) := L(t)$ where $\widehat{\beta} \in \widehat{L}(\widehat{t})$ and $(t, \widehat{t}) \in Z^{+}$, then the function f is well-defined. Moreover, let $t \in T_b$ be a (∇)-node such that $L_b(t) = \{\nabla\Psi_1, \dots, \nabla\Psi_i, l_1, \dots, l_j\}$. Then, we expand f to the formula χ_1 and χ_2 such that

$$\mathsf{anf}(u) \leq \chi_1 \leq \bigvee_{u \in C_b^{(k)}(t)} \mathsf{anf}(u) \leq \chi_2 \leq \left(\bigvee_{1 \leq k \leq i} \left(\bigvee_{u \in C_b^{(k)}(t)} \mathsf{anf}(u) \right) \right),$$

for every k where $1 \leq k \leq i$ and for every $u \in C_b^{(k)}(t)$. Now, we define $f(\chi_2)$ as

$$f(\chi_2) := \left\{ \bigvee \Psi_n \mid 1 \leq n \leq i \right\}.$$

Next, we note that for any $u \in C_b^{(k)}(t)$ there is a unique $\psi_k \in \Psi_k$ such that $\nabla\Psi_k$ is reduced to ψ_k. We denote such a ψ_k by $\mathsf{cor}(u)$. Suppose $\chi_1 = \bigvee_{u \in X^{(k)}} \mathsf{anf}(u)$ where $X^{(k)} \subseteq C_b^{(k)}(t)$. Then we define $f(\chi_1)$ as;

$$f(\chi_1) := \left\{ \bigvee \Psi_n \mid 1 \leq n \leq i,\ n \neq k \right\} \cup \left\{ \bigvee_{u \in X^{(k)}} \mathsf{cor}(u) \right\}.$$

Recalling Equation (8), the reason we designated the order of disjunction in $\mathsf{anf}(t)$ is that, in conjunction with above definition of f, we obtain the following useful property:

(Corresponding Property): Consider the section of the tableau which has the root labeled by

$$\left(\bigvee_{1 \leq k \leq i} \left(\bigvee_{u \in C_b^{(k)}(t)} \mathsf{anf}(u) \right) \right),$$

and every leaf labeled by some $\mathsf{anf}(u)$. Then, for any node u and its children v_1 and v_2 we have (i) $f(L(u)) = f(L(v_1)) = f(L(v_2))$ or, (ii) $f(L(u))$, $f(L(v_1))$ and $f(L(v_2))$ forming a ($\vee$)-rule.

Let us confirm the above property by observing a concrete example as depicted in Figure 3. In this example, the root and its children satisfy (i), and the child of the root and its children form a ($\vee$)-rule. Thus, (ii) is satisfied.

The function f will be used in the proof of Part 5 of Lemma 5.26.

Corollary 5.19. *For any well-named formula α, we can construct an automaton normal form $\mathsf{anf}(\alpha)$ which is semantically equivalent to α. Moreover, for any $x \in \mathsf{Free}(\alpha)$ which occurs only positively in α, it holds that $x \in \mathsf{Free}(\mathsf{anf}(\alpha))$ and x occurs only positively in $\mathsf{anf}(\alpha)$.*

Proof. This is an immediate consequence of Lemma 5.16 and Theorem 5.17. ☐

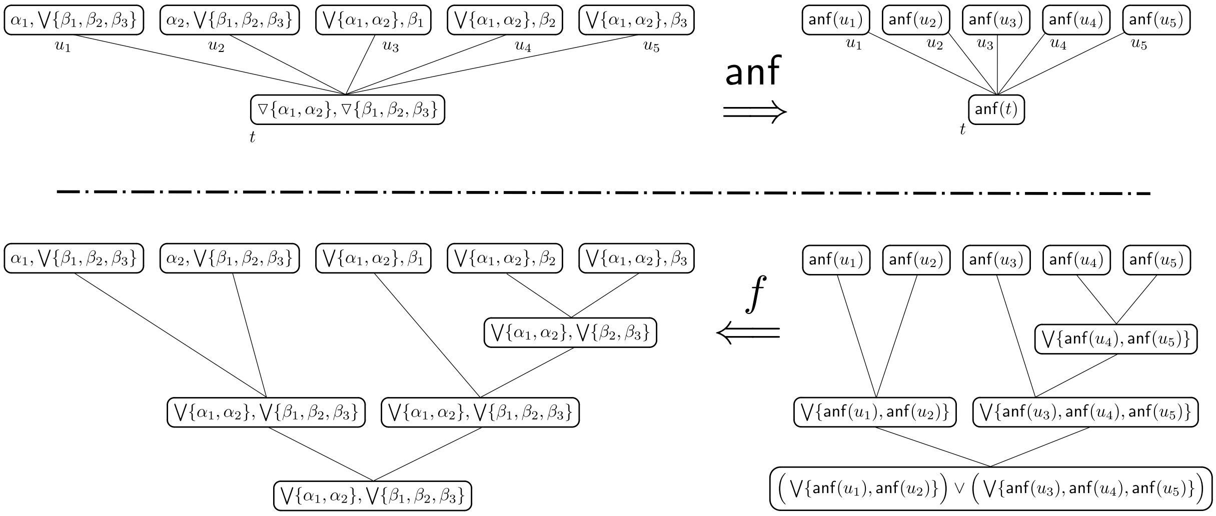

Figure 3: An example of the corresponding property.

5.3 Wide tableau

Definition 5.20 (Wide tableau). Let φ be a well-named formula. The rule of a *wide tableau* for φ is obtained by adding the following seven rules to the rule of tableau, which are collectively called the *wide rules*:

$$\frac{\Gamma}{\Gamma}\ (\epsilon_1) \qquad \frac{\Gamma \mid \Gamma}{\Gamma}\ (\epsilon_2)$$

$$\frac{\alpha, \alpha \vee \beta, \Gamma \mid \beta, \alpha \vee \beta, \Gamma}{\alpha \vee \beta, \Gamma}\ (\vee_w) \qquad \frac{\alpha, \beta, \alpha \wedge \beta, \Gamma}{\alpha \wedge \beta, \Gamma}\ (\wedge_w)$$

$$\frac{\varphi_x(x), \sigma_x x.\varphi_x(x), \Gamma}{\sigma_x x.\varphi_x(x), \Gamma}\ (\sigma_w) \qquad \frac{\varphi_x(x), x, \Gamma}{x, \Gamma}\ (\mathsf{Reg}_w)$$

$$\frac{\{\psi_k\} \cup \{\bigvee \Psi_n \mid n \in N_{\psi_k}\} \mid \text{For every } k \in \omega \text{ with } 1 \le k \le i \text{ and } \psi_k \in \Psi_k.}{\nabla\Psi_1, \ldots, \nabla\Psi_i, l_1, \ldots, l_j}\ (\nabla_w)$$

where in the (∇_w)-rule, $l_1, \ldots, l_j \in \mathsf{Lit}(\varphi)$ and, for each $\psi_k \in \bigcup_{1 \le k \le i} \Psi_k$ we have $N_{\psi_k} = \{n \in \omega \mid 1 \le n \le i,\ n \ne k\}$ or $N_{\psi_k} = \{n \in \omega \mid 1 \le n \le i\}$. Therefore, the premises of the (∇_w)-rule is, as with the (∇)-rule, equal to $\sum_{1 \le k \le i} |\Psi_k|$.

A *wide tableau* for φ (notation: $\mathcal{WT}_\varphi$) is the structure defined as a tableau for φ, but satisfying the following additional clause:

4. For any infinite branch π of $\mathcal{WT}_\varphi$, $\{n \in \omega \mid \pi[n]$ is (∇)-node or (∇_w)-node$\}$ is an infinite set.

Clause 4 restrains a branch that does not reach any modal node eternally by infinitely applying the wide rules except (∇_w)-rule.

Remark 5.21. A tableau can be considered a special case of the wide tableau, in which the wide-rules are not used. The concepts of modal and choice nodes as per Definition 5.5 naturally extend to the wide tableau. Let t be a node of some wide tableau and u be its child. Then, the trace function TR_{tu} as per Definition 5.6 is extended as follows:

- If t is a (ϵ_1)- or (ϵ_2)-node where t and u are labeled by Γ, then we set $\mathsf{TR}_{tu}(\gamma) := \{\gamma\}$ for every $\gamma \in \Gamma$.
- If t is a $(\vee_w)$-node where the rule applied between t and its children forms

$$\frac{\alpha, \alpha \vee \beta, \Gamma \mid \beta, \alpha \vee \beta, \Gamma}{\alpha \vee \beta, \Gamma}\ (\vee)$$

  then we set $\mathsf{TR}_{tu}(\gamma) := \{\gamma\}$ for every $\gamma \in \Gamma$. Furthermore, we set $\mathsf{TR}_{tu}(\alpha \vee \beta) := \{\alpha, \alpha \vee \beta\}$ when $L(u) = \{\alpha, \alpha \vee \beta\} \cup \Gamma$ and set $\mathsf{TR}_{tu}(\alpha \vee \beta) := \{\beta, \alpha \vee \beta\}$ when $L(u) = \{\beta, \alpha \vee \beta\} \cup \Gamma$.

- If t is a $(\wedge_w)$-node where the rule applied between t and its child forms

$$\frac{\alpha, \beta, \alpha \wedge \beta, \Gamma}{\alpha \wedge \beta, \Gamma} \ (\wedge_w)$$

then we set $\mathsf{TR}_{tu}(\gamma) := \{\gamma\}$ for every $\gamma \in \Gamma$, and we set $\mathsf{TR}_{tu}(\alpha \wedge \beta) := \{\alpha, \beta, \alpha \wedge \beta\}$.

- If t is a (σ_w)-node where the rule applied between t and its child forms

$$\frac{\varphi_x(x), \sigma_x x.\varphi_x(x), \Gamma}{\sigma_x x.\varphi_x(x), \Gamma} \ (\sigma_w)$$

then we set $\mathsf{TR}_{tu}(\gamma) := \{\gamma\}$ for every $\gamma \in \Gamma$, and we set $\mathsf{TR}_{tu}(\sigma_x x.\varphi_x(x)) := \{\varphi_x(x), \sigma_x x.\varphi_x(x)\}$.

- If t is a (Reg)-node where the rule applied between t and its child forms

$$\frac{\varphi_x(x), x, \Gamma}{x, \Gamma} \ (\mathsf{Reg}_w)$$

then we set $\mathsf{TR}_{tu}(\gamma) := \{\gamma\}$ for every $\gamma \in \Gamma$, and we set $\mathsf{TR}_{tu}(x) := \{\varphi_x(x), x\}$.

- If t is a (∇_w)-node where the rule applied between t and its children forms

$$\frac{\{\psi_k\} \cup \{\bigvee \Psi_n \mid n \in N_{\psi_k}\} \ \mid \ 1 \leq k \leq i, \ \psi_k \in \Psi_k.}{\nabla\Psi_1, \ldots, \nabla\Psi_i, l_1, \ldots, l_j} \ (\nabla_w)$$

Moreover, suppose u is labeled by $\{\psi_k\} \cup \{\bigvee \Psi_n \mid n \in N_{\psi_k}\}$. Then, we set $\mathsf{TR}_{tu}(\nabla\Psi_k) := \{\psi_k\}$ when $N_{\psi_k} = \{n \in \omega \mid 1 \leq n \leq i, \ n \neq k\}$, and we set $\mathsf{TR}_{tu}(\nabla\Psi_k) := \{\psi_k, \bigvee \Psi_k\}$ when $N_{\psi_k} = \{n \in \omega \mid 1 \leq n \leq i\}$. We set $\mathsf{TR}_{tu}(\nabla\Psi_n) := \{\bigvee \Psi_n\}$ for every $n \in N_{\psi_k} \setminus \{k\}$, and set $\mathsf{TR}_{tu}(l_n) := \emptyset$ for every $n \leq j$.

Under this extended definition of the trace, the automaton $\mathcal{A}_\varphi$ of Lemma 5.7 and the bisimulation of Definition 5.14 can also be naturally extended to the wide tableau. Thus, we apply these concepts and results freely to this new structure.

Definition 5.22 (**Inserted trace).** Let $\mathcal{WT}_\varphi = (T, C, r, L)$ be a wide tableau for some well-named formula φ. Let π be a finite or infinite branch of $\mathcal{WT}$ and let tr be a trace on π. For technical reasons, we will need an *inserted trace* (denotation: tr^+) for each trace tr which is constructed by the following procedure (†) (see also Figure 4);

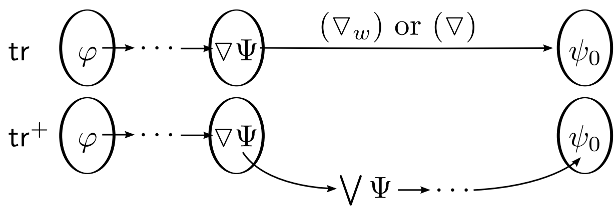

Figure 4: An inserted trace.

(†): Suppose $\Psi = \{\psi_0, \psi_1, \ldots, \psi_k\}$ and that $\pi[n]$ is a (∇)- or (∇_w)-node in which $\mathsf{tr}[n] = \nabla\Psi$ is reduced into $\mathsf{tr}[n+1] = \psi_0$. Then, we insert the sequence

$$\langle \bigvee \Psi, \bigvee(\Psi \setminus \{\ \psi_1\}), \bigvee(\Psi \setminus \{\ \psi_1, \psi_2\}), \ldots, \bigvee\{\psi_0, \psi_{k-1}, \psi_k\}, \bigvee\{\psi_0, \psi_k\}\rangle$$

between $\mathsf{tr}[n]$ and $\mathsf{tr}[n+1]$.

Note that tr is even if and only if tr^+ is even because inserted formulas are all $\vee$-formulas and, thus, the priorities of these formulas are equal to 0 (recall Equation (6)). The set of inserted traces $\mathsf{TR}^+(\pi)$ and the set of factors of inserted traces $\mathsf{TR}^+(\pi[n, m])$ or $\mathsf{TR}^+(\pi[n], \pi[m])$ are defined similarly.

Definition 5.23 (Tableau consequence). Let $\mathcal{WT}_\alpha = (T, C, r, L)$ and $\mathcal{WT}_\beta = (T', C', r', L')$ be two wide tableaux for some well-named formula α and β. Let T_m and T'_m be the set of modal nodes of $\mathcal{WT}_\alpha$ and $\mathcal{WT}_\beta$, and let T_c and T'_c be the set of choice nodes of $\mathcal{WT}_\alpha$ and $\mathcal{WT}_\beta$, respectively. Then $\mathcal{WT}_\beta$ is called a *tableau consequence* of $\mathcal{WT}_\alpha$ (notation: $\mathcal{WT}_\alpha \rightharpoonup \mathcal{WT}_\beta$) if there exists a binary relation $Z \subseteq (T_m \times T'_m) \cup (T_c \times T'_c)$ satisfying the following six conditions (here, the condition of the tableau consequence is similar to the condition of tableau bisimulation so we have illustrated the differences between these two conditions using underlines):

Root condition: $(r, r') \in Z$.

Prop condition: For any $t \in T_m$ and $t' \in T'_m$, if $(t, t') \in Z$, then

$$(L(t) \cap \mathsf{Lit}(\alpha)) \setminus \mathsf{Top} \underset{\sim}{\supseteq} (L'(t') \cap \mathsf{Lit}(\beta)) \setminus \mathsf{Top}.$$

Consequently, $L(t)$ is consistent only if $L'(t')$ is consistent.

Forth condition on modal nodes: Take $t, u \in T_m$ and $t' \in T'_m$ arbitrarily. If $(t, t') \in Z$ and u is a next modal node of t, then $C'(t') = \emptyset$ or there exists $u' \in T'_m$ which is a next modal node of t' such that $(u, u') \in Z$.

Back condition on modal nodes: Take $t \in T_m$, $t' \in T'_m$ and $u' \in T'_c$ arbitrarily. If $(t, t') \in Z$ and $u' \in C'(t')$, then $C(t) = \emptyset$ or there exists $u \in C(t)$ such that $(u, u') \in Z$.

Forth condition on choice nodes: Take $u \in T_c$, $t \in T_m$ and $u' \in T'_c$ arbitrarily. If $(u, u') \in Z$ and t is near u, then there exists $t' \in T'_m$ such that $(t, t') \in Z$ and t' is near u'.

Back condition on choice nodes: No condition.

Parity condition: Let π and π' be infinite branches of $\mathcal{WT}_\alpha$ and $\mathcal{WT}_\beta$ respectively. If π and π' are associated with each other, then π is even only if π' is even.

A relation Z which satisfies the above six conditions called *tableau consequence relation* from $\mathcal{WT}_\alpha$ to $\mathcal{WT}_\beta$.

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be tableaux mentioned in Remark 5.15, then they are not bisimilar. However, we can assume that $\mathcal{T}_2 \rightharpoonup \mathcal{T}_1$. Suppose t is a node of some tableau labeled by $\{\gamma\} \cup \Gamma$ and, u is a its child labeled by $\{\gamma'\} \cup \Gamma$. Then, there exists two possibilities; $\gamma' \in \Gamma$ or $\gamma' \notin \Gamma$. We say a collision occurred between t and u if $\gamma' \in \Gamma$. In Remark 5.15, we can find collisions in $\mathcal{T}_1$ but cannot in $\mathcal{T}_2$. In general, if we construct a tableau $\mathcal{T}_\varphi$ for a given formula φ so that collisions occur as many as possible, then, we have $\mathcal{WT}_\varphi \rightharpoonup \mathcal{T}_\varphi$ for any wide tableau $\mathcal{WT}_\varphi$ for φ. To denote this fact correctly, we introduce the following definition and lemma.

Definition 5.24 (Narrow tableau). A well-named formula φ and a set $\Gamma \subseteq \mathsf{Sub}(\varphi)$ are given. For a formula $\gamma \in \Gamma$, a closure of γ (denotation: $\mathsf{cl}(\gamma)$) is defined as follows:

- $\gamma \in \mathsf{cl}(\gamma)$.
- If $\alpha \circ \beta \in \mathsf{cl}(\gamma)$, then $\alpha, \beta \in \mathsf{cl}(\gamma)$ where $\circ \in \{\vee, \wedge\}$.
- If $\sigma_x x.\varphi_x(x) \in \mathsf{cl}(\gamma)$, then $\varphi_x(x) \in \mathsf{cl}(\gamma)$.
- If $x \in \mathsf{cl}(\gamma) \cap \mathsf{Bound}(\varphi)$, then $\varphi_x(x) \in \mathsf{cl}(\gamma)$.

In other words, $\mathsf{cl}(\gamma)$ is a set of all formulas δ such that for any tableau $\mathcal{T}_\varphi = (T, C, r, L)$ and its node $t \in T$, if $\gamma \in L(t)$, then, there is a descendant $u \in C^*(t)$ near t and a trace tr on the C-sequence from t to u where $\mathsf{tr}[1] = \gamma$ and $\mathsf{tr}[|\mathsf{tr}|] = \delta$. We say γ is *reducible* in Γ if, for any $\gamma' \in \Gamma \setminus \{\gamma\}$, we have $\gamma \notin \mathsf{cl}(\gamma')$. A tableau $\mathcal{T}_\varphi = (T, C, r, L)$ is said *narrow* if for any node $t \in T$ which is not modal, the reduced formula $\gamma \in L(t)$ between t and its children is reducible in $L(t)$.

Lemma 5.25. *For any well-named formula φ, we can construct a narrow tableau for φ.*

Proof. Let φ be a well-named formula. Then, it is enough to show that for any $\Gamma \subseteq \mathsf{Sub}(\varphi)$ which is not modal, there exists a reducible formula $\gamma \in \Gamma$. Suppose, moving toward a contradiction, that there exists $\Gamma \subseteq \mathsf{Sub}(\varphi)$ which is not modal and does not include any reducible formula. Take a formula $\gamma_1 \in \Gamma$ such that $\mathsf{cl}(\gamma_1) \supsetneq \{\gamma_1\}$. Since γ_1 is not reducible in Γ, there exists $\gamma_2 \in \Gamma \setminus \{\gamma_1\}$ such that $\gamma_1 \in \mathsf{cl}(\gamma_2)$. Since γ_2 is not reducible in Γ, there exists $\gamma_3 \in \Gamma \setminus \{\gamma_2\}$ such that $\gamma_2 \in \mathsf{cl}(\gamma_3)$. And so forth, we obtain the sequence $\langle \gamma_n \mid n \in \omega \setminus \{0\} \rangle$ such that $\gamma_{n+1} \in \Gamma \setminus \{\gamma_n\}$ and $\gamma_n \in \mathsf{cl}(\gamma_{n+1})$ for any $n \in \omega \setminus \{0\}$. Since $|\Gamma|$ is finite, there exists $i, j \in \omega$ such that $1 \leq i < j$ and $\gamma_i = \gamma_j$. Consider the tableau $\mathcal{T}_\varphi = (T, C, r, L)$ and its node $t \in T$ such that $\gamma_j \in L(t)$. Then, from the definition of the closure cl, there exists a trace tr on π such that:

($\heartsuit$) π is a finite C-sequence starting at t where (∇)-rule nor (∇_w)-rule do not applied between π.

($\clubsuit$) $\mathsf{tr}[1] = \mathsf{tr}[|\mathsf{tr}|] = \gamma_j$.

On the other hand, since φ is well-named, for any bound variable $x \in \mathsf{Bound}(\varphi)$, x is in the scope of some modal operator (cover modality) in $\varphi_x(x)$. Thus we have:

($\spadesuit$) For any trace tr on π, if ($\clubsuit$) is satisfied, then π includes a (∇)-node or (∇_w)-node.

($\heartsuit$) and ($\spadesuit$) contradict each other. ☐

The next lemma states some basic properties of the tableau consequence.

Lemma 5.26. *Let α, β, γ and $\varphi(x)$ be well-named formulas where x appears only positively and in the scope of some modality in $\varphi(x)$. Then, we have:*

1. *If $\mathcal{WT}_\alpha \rightleftharpoons \mathcal{WT}_\beta$, then $\mathcal{WT}_\alpha \rightharpoonup \mathcal{WT}_\beta$.*
2. *If $\mathcal{WT}_\alpha \rightharpoonup \mathcal{WT}_\beta$ and $\mathcal{WT}_\beta \rightharpoonup \mathcal{WT}_\gamma$, then $\mathcal{WT}_\alpha \rightharpoonup \mathcal{WT}_\gamma$.*
3. *If $\mathcal{T}_\alpha$ is narrow, then, for any wide tableau $\mathcal{WT}_\alpha$ we have $\mathcal{WT}_\alpha \rightharpoonup \mathcal{T}_\alpha$.*
4. *For any tableau $\mathcal{T}_{\varphi(\mu\vec{x}.\varphi(\vec{x}))}$, there exists a wide tableau $\mathcal{WT}_{\mu\vec{x}.\varphi(\vec{x})}$ such that $\mathcal{T}_{\varphi(\mu\vec{x}.\varphi(\vec{x}))} \rightleftharpoons \mathcal{WT}_{\mu\vec{x}.\varphi(\vec{x})}$.*
5. *For any tableau $\mathcal{T}_{\varphi(\mathsf{anf}(\alpha))}$, there exists a wide tableau $\mathcal{WT}_{\varphi(\alpha)}$ such that $\mathcal{T}_{\varphi(\mathsf{anf}(\alpha))} \rightleftharpoons \mathcal{WT}_{\varphi(\alpha)}$.*

Proof. [Part 1] Suppose $\mathcal{WT}_\alpha \rightleftharpoons \mathcal{WT}_\beta$. Then there exists a tableau bisimulation Z from $\mathcal{WT}_\alpha$ to $\mathcal{WT}_\beta$. It is easily checked that Z satisfies the conditions of the tableau consequence relation from $\mathcal{WT}_\alpha$ to $\mathcal{WT}_\beta$ and, thus, $\mathcal{WT}_\alpha \rightharpoonup \mathcal{WT}_\beta$.

[Part 2] Suppose $\mathcal{WT}_\alpha \rightharpoonup \mathcal{WT}_\beta$ and $\mathcal{WT}_\beta \rightharpoonup \mathcal{WT}_\gamma$. Then, there is a tableau consequence relation, Z, from $\mathcal{WT}_\alpha$ to $\mathcal{WT}_\beta$ and there is a tableau consequence relation, Z', from $\mathcal{WT}_\beta$ to $\mathcal{WT}_\gamma$. The composition $ZZ' := \{(t, t'') \mid (t, t') \in Z,\ (t', t'') \in Z'\}$ is a tableau consequence relation from $\mathcal{WT}_\alpha$ to $\mathcal{WT}_\gamma$ and, thus $\mathcal{WT}_\alpha \rightharpoonup \mathcal{WT}_\gamma$.

[Part 3] Let T_c and T'_c be the sets of choice nodes of $\mathcal{WT}_\alpha$ and $\mathcal{T}_\alpha$, and let T_m and T'_m be the sets of modal nodes of $\mathcal{WT}_\alpha$ and $\mathcal{T}_\alpha$, respectively. The tableau consequence relation Z is constructed inductively in a bottom-up fashion. Our construction of Z satisfies the following additional property:

(†) For any $t \in T_c \cup T_m$ and $t' \in T'_c \cup T'_m$, if $(t, t') \in Z$ then $L(t) \supseteq L'(t')$.

For the base step, add (r, r') into Z. This expansion indeed satisfies (†), since $L(r) = L'(r') = \{\alpha\}$. The inductive step is divided into two cases.

For the first case, suppose that $u \in T_c$ and $u' \in T'_c$ satisfies $(u, u') \in Z$ and (†). From the facts $L(u) \supseteq L'(u')$ and that $\mathcal{T}_\alpha$ is narrow, for any $t \in T_m$ which is near u, we can find $t' \in T'_m$ which is near u' such that $\mathsf{TR}[u, t] \Supset \mathsf{TR}[u', t']$. We add such pairs (t, t') into Z; this expansion indeed preserves (†). Note that it is possible that, although $L(u) = L'(u')$, our extension yields $L(t) \supsetneq L'(t')$ due to collisions and the $(\vee_w)$-rule. For example, consider a section of a wide tableau and a tableau as depicted in Figure 5. In this example, if $(u, u') \in Z$, we must extend it so that Z includes

$$\{(t_1, t'_1), (t_2, t'_1), (t_3, t'_1), (t_2, t'_2), (t_3, t'_2), (t_4, t'_2)\}$$

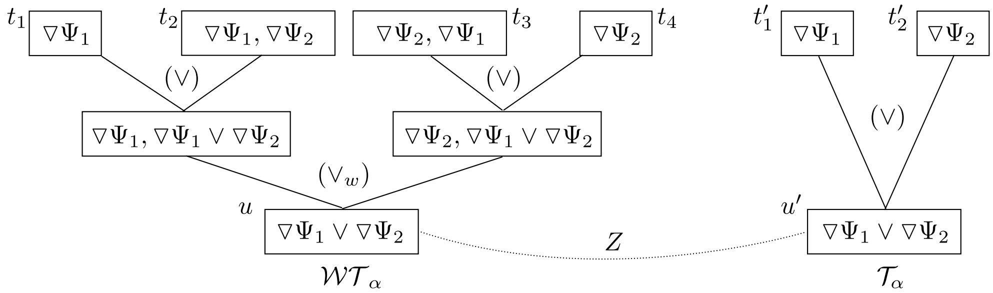

Figure 5: An extension of the tableau consequence relation.

because of, for example,

$$\begin{array}{rcl} \mathsf{TR}[u, t_2] & = & \{\langle \nabla\Psi_1 \vee \nabla\Psi_2, \quad \nabla\Psi_1, \quad \nabla\Psi_1\rangle, \\ & & \langle \nabla\Psi_1 \vee \nabla\Psi_2, \quad \nabla\Psi_1 \vee \nabla\Psi_2, \quad \nabla\Psi_2\rangle\} \\ & \Supset & \{\langle \nabla\Psi_1 \vee \nabla\Psi_2, \quad \nabla\Psi_1\rangle\} \\ & = & \mathsf{TR}[u', t'_1]. \end{array}$$

Thus, we have $(t_2, t'_1) \in Z$. Consequently, although $L(u) = L'(u') = \{\nabla\Psi_1 \vee \nabla\Psi_2\}$, we have $L(t_2) = \{\nabla\Psi_1, \nabla\Psi_2\} \supsetneq \{\nabla\Psi_1\} = L'(t'_1)$.

For the second case, suppose that $t \in T_m$ and $t' \in T'_m$ satisfy $(t, t') \in Z$ and (†). Let

$$L(t) = \nabla\Psi_1, \ldots, \nabla\Psi_a, \nabla\Psi_{a+1}, \ldots, \nabla\Psi_b, l_1, \ldots, l_c, l_{c+1}, \ldots, l_d, \tag{10}$$

$$L'(t') = \nabla\Psi_1, \ldots, \nabla\Psi_a, \qquad l_1, \ldots, l_c, \tag{11}$$

with $0 \le a, b, c, d$. If $a = 0$, then we halt the expansion of Z from (t, t'). This halting procedure does not conflict with the forth and back conditions on modal nodes t and t' since $C'(t') = \emptyset$. Similarly, if $\{l_1, \ldots, l_d\}$ is inconsistent, then we halt the expansion of Z from (t, t'). This halting procedure does not conflict with the forth and back conditions on modal nodes t and t' since $C(t) = \emptyset$. Suppose $a > 0$ and $\{l_1, \ldots, l_d\}$ is consistent. Then, for the back condition on modal nodes, for any $u' \in C'(t')$, we must find $u \in C(t)$ such that $(u, u') \in Z$. For $u' \in C'(t')$ which is labeled by $\{\psi_k\} \cup \{\bigvee \Psi_n \mid n \in N'_{\psi_k}\}$, we add pairs (u, u') into Z where $u \in C(t)$ is labeled by $\{\psi_k\} \cup \{\bigvee \Psi_n \mid n \in N_{\psi_k}\}$. This expansion clearly preserves Condition (†). For the forth condition on modal nodes, for any u which is a next modal node of t we must find u' which is a next modal node of t' such that $(u, u') \in Z$. From (10), (11) and the fact that $\mathcal{T}_\alpha$ is narrow, for any u near t, there exists u' near t' such that $\mathsf{TR}^+[t, u] \Supset \mathsf{TR}^+[t', u']$. We add such pairs (u, u') into Z. Again, this expansion preserves (†).

Finally, we must prove that the relation Z constructed above satisfies the parity condition. Let π and π' be infinite branches of $\mathcal{WT}_\alpha$ and $\mathcal{T}_\alpha$, respectively, such that π and π' are associated with each other. Then, by the construction of Z, we can assume that

$$\mathsf{TR}^+(\pi) \Supset \mathsf{TR}^+(\pi') \tag{12}$$

If π' is *not* even, then there exists an odd trace tr on π'. From (12), we can assume that $\mathsf{TR}^+(\pi)$ includes tr^+ and, thus, there exists an odd trace on π (this is because, remember that tr is even if and only if tr^+ is even). This means π is also *not* even and, therefore, the parity condition is indeed satisfied.

Part 4 First, recall Remark 2.11. Since $\varphi(x)$ is well-named, we can assume that $\varphi(\mu\vec{x}.\varphi(\vec{x}))$ is an abbreviation of

$$\varphi(\mu\vec{x}_1.\varphi(\vec{x}_1), \ldots, \mu\vec{x}_k.\varphi(\vec{x}_k))$$

where $\varphi(x) = \varphi(x_1, \ldots, x_k)[x_1/x, \ldots, x_k/x]$, $x \notin \mathsf{Free}(\varphi(x_1, \ldots, x_k))$ and $\mu\vec{x}_i.\varphi(\vec{x_i}) = \mu x_i^{(1)} \ldots \mu x_i^{(k)}.\varphi(x_i^{(1)}, \ldots, x_i^{(k)})$ with $1 \le i \le k$ are appropriate renaming formulas of $\mu\vec{x}.\varphi(\vec{x})$ so that Equations (1) through (5) are satisfied. Then, we can divide $\mathsf{Sub}(\varphi(\mu\vec{x}.\varphi(\vec{x}))$ into the following three sets of formulas, each of them pairwise disjoint;

$$\mathsf{Sub}_1 := \big\{\alpha(\mu\vec{x}_1.\varphi(\vec{x}_1), \ldots, \mu\vec{x}_k.\varphi(\vec{x}_k)) \mid \alpha(\vec{x}) \in \mathsf{Sub}(\mu\vec{x}.\varphi(\vec{x}))\big\} \setminus \mathsf{Sub}_3$$

$$\mathsf{Sub}_2 := \bigcup_{1 \le i \le k} \mathsf{Sub}(\mu\vec{x}_i.\varphi(\vec{x}_i)) \setminus \Big(\big\{\mu\vec{x}_1.\varphi(\vec{x}_1), \ldots, \mu\vec{x}_k.\varphi(\vec{x}_k))\big\} \cup \mathsf{Sub}_3\Big)$$

$$\mathsf{Sub}_3 := \{\psi \in \mathsf{Sub}(\varphi(\mu\vec{x}.\varphi(\vec{x}))) \mid \psi \text{ does not contain any bound variable.}\}$$

Next, we define the function $f : \mathsf{Sub}(\varphi(\mu\vec{x}.\varphi(\vec{x}))) \to \mathsf{Sub}(\mu\vec{x}.\varphi(\vec{x}))$ by

$$f(\psi) := \begin{cases} \alpha(\vec{x}) & \text{if } \psi = \alpha(\mu\vec{x}_1.\varphi(\vec{x_1}), \dots, \mu\vec{x}_k.\varphi(\vec{x_k})) \in \mathsf{Sub}_1, \\ \beta(\vec{x}) & \text{if } \psi = \beta(\vec{x}_i) \in \mathsf{Sub}_2 \text{ with } 1 \le i \le k, \\ \psi & \text{if } \psi \in \mathsf{Sub}_3. \end{cases}$$

Let $\mathcal{T}_{\varphi(\mu\vec{x}.\varphi(\vec{x}))} = (T, C, r, L)$ be a tableau for $\varphi(\mu\vec{x}.\varphi(\vec{x}))$. Consider the structure

$$\mathcal{WT}_{\mu\vec{x}.\varphi(\vec{x})} = (T \uplus \{r_1, \dots, r_k\}, C \uplus \{(r_n, r_{n+1}), (r_k, r) \mid 1 \le n < k\}, r_1, L')$$

where $L'(r_n) := \{\mu x_n. \dots \mu x_k.\varphi(\vec{x})\}$ with $1 \le n \le k$ and $L'(t) := f(L(t))$ for any $t \in T$. Then, we can assume $\mathcal{WT}_{\mu\vec{x}.\varphi(\vec{x})}$ is a wide tableau for $\mu\vec{x}.\varphi(\vec{x})$. Note that, in general, wide rules are necessary in $\mathcal{WT}_{\mu\vec{x}.\varphi(\vec{x})}$; for example, consider a part of $\mathcal{T}_{\varphi(\mu\vec{x}.\varphi(\vec{x}))}$ and the corresponding part of $\mathcal{WT}_{\mu\vec{x}.\varphi(\vec{x})}$ depicted in Figure 6. In this example, we assume that considering the label of a node includes

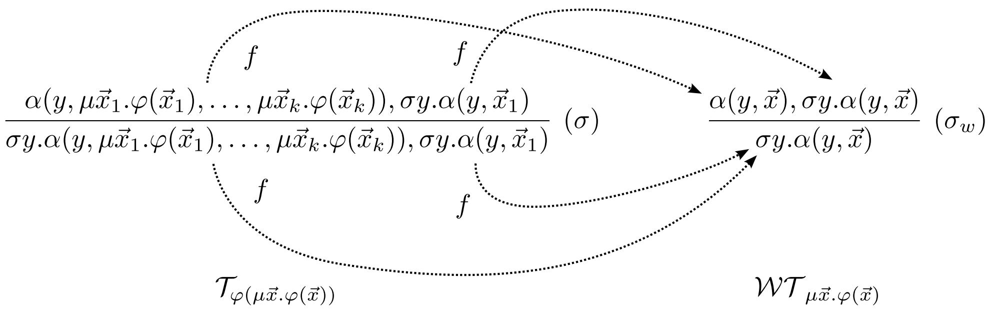

Figure 6: An initial example of a corresponding wide tableau.

$$\psi_1 := \sigma y.\alpha(y, \mu\vec{x}_1.\varphi(\vec{x}_1), \dots, \mu\vec{x}_k.\varphi(\vec{x}_k)) \in \mathsf{Sub}_1$$
$$\psi_2 := \sigma y.\alpha(y, \vec{x}_1) \in \mathsf{Sub}_2$$

where $f(\psi_1) = f(\psi_2) = \sigma y.\alpha(y, \vec{x})$. If we reduce ψ_1, then the corresponding label of the node on $\mathcal{WT}_{\mu\vec{x}.\varphi(\vec{x})}$ includes $\alpha(y, \vec{x})$ and $\sigma y.\alpha(y, \vec{x})$. Therefore, this case requires the (σ_w)-rule.

Take an infinite trace tr of $\mathcal{T}_{\varphi(\mu\vec{x}.\varphi(\vec{x}))}$ arbitrarily. Then, from the definition of f, we have;

(‡): tr is even if and only if $\vec{f}(\mathsf{tr})$ is even.

Set $Z := \{(r, r_1), (t, t) \mid t \in (T_m \cup T_c) \setminus \{r\}\}$, where $T_m \subseteq T$ is the set of modal nodes and $T_c \subseteq T$ is the set of choice nodes. This relation Z satisfies the conditions of tableau bisimulation; we only have to confirm the parity condition since all the other conditions are obviously satisfied. Let π be an infinite branch of $\mathcal{T}_{\varphi(\mu\vec{x}.\varphi(\vec{x}))}$ and let π' be an associated infinite branch of $\mathcal{WT}_{\mu\vec{x}.\varphi(\vec{x})}$. Then, from the construction of $\mathcal{WT}_{\mu\vec{x}.\varphi(\vec{x})}$ and Z, we can assume that $\pi[n] = \pi'[n+k]$ for every $n \in \omega \setminus \{0, 1\}$. If π is not even, then there exists a trace tr on π which is not even. Consider the sequence $\langle \mu\vec{x}.\varphi(\vec{x}), \dots, \mu x_k.\varphi(\vec{x})\rangle \vec{f}(\mathsf{tr})$. From (‡), we can assume that this sequence is a trace on π', which is also not even. Therefore π' is not even. Conversely, suppose that π' is not even. Then, there exists a trace tr' on π' which is not even. Take a trace tr on π such that $\langle \mu\vec{x}.\varphi(\vec{x}), \dots, \mu x_k.\varphi(\vec{x})\rangle \vec{f}(\mathsf{tr}) = \mathsf{tr}'$. Then, tr is also not even and, therefore, π is not even. The above implies the parity condition of Z.

Part 5 First, as in the proof of Part 4, we divide $\mathsf{Sub}(\varphi(\mathsf{anf}(\alpha)))$ into three sets of formulas, each of them pairwise disjoint;

$$\mathsf{Sub}_1 := \{\beta(\mathsf{anf}(\alpha)_1, \dots, \mathsf{anf}(\alpha)_k) \mid \beta(x_1, \dots, x_k) \in \mathsf{Sub}(\varphi(\vec{x}))\} \setminus \mathsf{Sub}_3$$
$$\mathsf{Sub}_2 := \bigcup_{1 \le i \le k} \mathsf{Sub}(\mathsf{anf}(\alpha)_i) \setminus \Big(\big\{\mathsf{anf}(\alpha)_1, \dots, \mathsf{anf}(\alpha)_k\big\} \cup \mathsf{Sub}_3\Big)$$
$$\mathsf{Sub}_3 := \{\psi \in \mathsf{Sub}(\varphi(\mathsf{anf}(\alpha))) \mid \psi \text{ does not contain any bound variable.}\}$$

where $\varphi(x) = \varphi(x_1, \dots, x_k)[x_1/x, \dots, x_k/x]$, $x \notin \mathsf{Free}(\varphi(x_1, \dots, x_k))$ and $\mathsf{anf}(\alpha)_i$ with $1 \le i \le k$ are appropriate renaming formulas of $\mathsf{anf}(\alpha)$. Recall Remark 5.18; there we had given the partial function

$f : \mathsf{Sub}(\mathsf{anf}(\alpha)) \to \mathcal{P}(\mathsf{Sub}(\alpha))$. We define the function $f^+ : \mathsf{Sub}(\varphi(\mathsf{anf}(\alpha))) \to \mathcal{P}(\mathsf{Sub}(\varphi(\alpha)))$ by expanding f as follows;

$$f^+(\psi) := \begin{cases} \{\beta(\alpha_1, \dots, \alpha_k)\} & \text{if } \psi = \beta(\mathsf{anf}(\alpha)_1, \dots, \mathsf{anf}(\alpha)_k) \in \mathsf{Sub}_1, \\ f(\widehat{\gamma}) & \text{if } \psi = \widehat{\gamma}_i \in \mathsf{Sub}_2 \text{ with } 1 \le i \le k, \\ \{\psi\} & \text{if } \psi \in \mathsf{Sub}_3. \end{cases}$$

where α_i with $1 \le i \le k$ are appropriate renaming formulas of α and $\widehat{\gamma}_i$ with $1 \le i \le k$ are appropriate renaming formula of $\widehat{\gamma} \in \mathsf{Sub}(\mathsf{anf}(\alpha))$. Let $\mathcal{T}_{\varphi(\mathsf{anf}(\alpha))} = (T, C, r, L)$ be a tableau for $\varphi(\mathsf{anf}(\alpha))$. Then, we can assume the corresponding structure $\mathcal{WT}_{\varphi(\alpha)} := (T, C, r, f^+ \circ L)$ is a wide tableau for $\varphi(\alpha)$. Note that the wide rules $(\wedge_w)$, $(\vee_w)$, (σ_w), (Reg_w) and (∇_w) are needed when we reduce χ_1 where the node under consideration includes χ_1 and χ_2 such that $f^+(\chi_1) \cap f^+(\chi_2) \neq \emptyset$. We observe this fact by confirming a constructed example depicted in Figure 7. In this example, the node of $\mathcal{T}_{\varphi(\mathsf{anf}(\alpha))}$ under consideration is

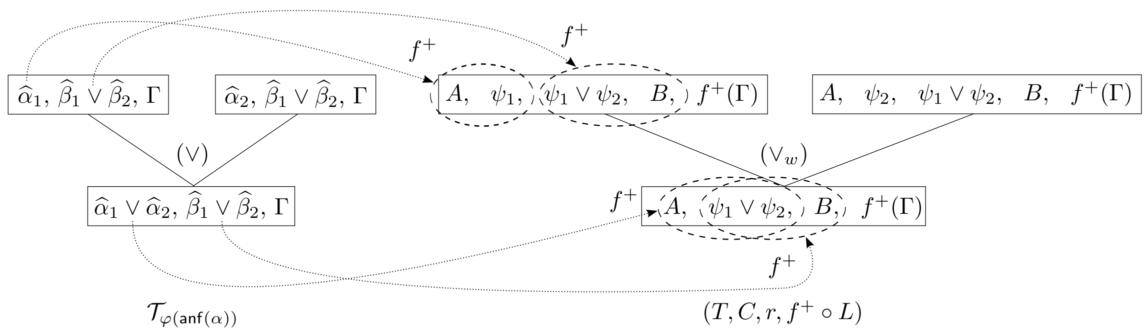

Figure 7: A second example of a corresponding wide tableau.

a $(\vee)$-node which is labeled by $\{\widehat{\alpha}_1 \vee \widehat{\alpha}_2, \widehat{\beta}_1 \vee \widehat{\beta}_2\} \cup \Gamma$ where $\widehat{\alpha}_1 \vee \widehat{\alpha}_2, \widehat{\beta}_1 \vee \widehat{\beta}_2 \in \mathsf{Sub}_2$ such that

$$\begin{aligned} f^+(\widehat{\alpha}_1 \vee \widehat{\alpha}_2) &= A \cup \{\psi_1 \vee \psi_2\} \\ f^+(\widehat{\beta}_1 \vee \widehat{\beta}_2) &= B \cup \{\psi_1 \vee \psi_2\} \\ f^+(\widehat{\alpha}_1) &= A \cup \{\psi_1\} \\ f^+(\widehat{\alpha}_2) &= A \cup \{\psi_2\} \end{aligned}$$

Thus, the corresponding labels $f^+ \circ L$ of such nodes form the $(\vee_w)$-rule. Moreover, note that the wide rules (ϵ_1) and (ϵ_2) are needed when we reduce χ_1 to χ_2 such that $f^+(\chi_1) = f^+(\chi_2)$.

Consider the relation $Z := \{(t, t) \mid t \in T_m \cup T_c\}$ where T_m is the set of modal nodes, and T_c is the set of choice nodes of $\mathcal{T}_{\varphi(\mathsf{anf}(\alpha))}$. To complete the proof, we have to show that Z is a bisimulation relation from $\mathcal{T}_{\varphi(\mathsf{anf}(\alpha))}$ to $\mathcal{WT}_{\varphi(\alpha)}$. It is obvious that Z satisfies the root condition, prop condition, forth conditions and back conditions. Therefore we only have to confirm the parity condition of Z. Let π be an infinite branch of $\mathcal{T}_{\varphi(\mathsf{anf}(\alpha))}$. We divide the set of traces $\mathsf{TR}(\pi)$ of $\mathcal{T}_{\varphi(\mathsf{anf}(\alpha))}$ into two sets; $\mathsf{TR}_1(\pi)$ consists of all traces tr such that $\mathsf{tr}[n] \in \mathsf{Sub}_1$ for every $n \in \omega$, and $\mathsf{TR}_2(\pi)$ consists of all traces tr such that $\mathsf{tr}[n] \in \mathsf{Sub}_2$ for some $n \in \omega$. Then $\vec{f^+}(\mathsf{TR}_1(\pi)) \cup \vec{f^+}(\mathsf{TR}_2(\pi))$ is the set of all traces of $\mathcal{WT}_{\varphi(\alpha)}$ on π. Since

$$\Omega_{\varphi(\alpha)}(\beta(\alpha)) = \Omega_{\varphi(\mathsf{anf}(\alpha))}(\beta(\mathsf{anf}(\alpha)_1, \dots, \mathsf{anf}(\alpha)_k)) \pmod 2$$

for any $\beta(\mathsf{anf}(\alpha)_1, \dots, \mathsf{anf}(\alpha)_k) \in \mathsf{Sub}_1$, we have

($\spadesuit$) : $\mathsf{TR}_1(\pi)$ includes an odd trace if and only if $\vec{f^+}(\mathsf{TR}_1(\pi))$ includes an odd trace.

On the other hand, for any $\mathsf{tr} \in \mathsf{TR}_2(\pi)$, from the construction of f^+, we have;

($\heartsuit$) : tr is odd if and only if $\vec{f^+}(\mathsf{tr})$ includes an odd trace.

From ($\spadesuit$) and ($\heartsuit$), we have that $\mathsf{TR}(\pi)$ is even if and only if $\vec{f^+}(\mathsf{TR}(\pi))$ is even, and so the parity condition is indeed satisfied. Therefore, Part 5 of the Lemma is true. ☐

Corollary 5.27. *Let $\widehat{\alpha}(x)$ be an automaton normal form in which $x \in \mathsf{Free}(\widehat{\alpha}(x))$ appears only positively. Set $\widehat{\varphi} := \mathsf{anf}(\mu\vec{x}.\widehat{\alpha}(\vec{x}))$. Then we have $\mathcal{T}_{\widehat{\alpha}(\widehat{\varphi})} \rightharpoonup \mathcal{T}_{\widehat{\varphi}}$.*

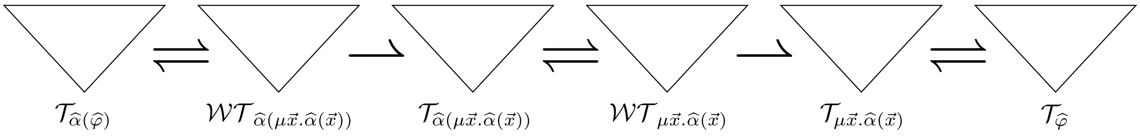

Figure 8: The plan for the proof of the corollary.

Proof. This corollary is proved using four wide tableaux; Figure 8 depicts the plan of the proof. First, we have $\mathcal{T}_{\widehat{\alpha}(\widehat{\varphi})} \rightleftharpoons \mathcal{WT}_{\widehat{\alpha}(\mu\vec{x}.\widehat{\alpha}(\vec{x}))}$ from Part 5 of Lemma 5.26. Second, take a narrow tableau $\mathcal{T}_{\widehat{\alpha}(\mu\vec{x}.\widehat{\alpha}(\vec{x}))}$, then, we have $\mathcal{WT}_{\widehat{\alpha}(\mu\vec{x}.\widehat{\alpha}(\vec{x}))} \rightharpoonup \mathcal{T}_{\widehat{\alpha}(\mu\vec{x}.\widehat{\alpha}(\vec{x}))}$ from Part 3 of Lemma 5.26. Third, we have $\mathcal{T}_{\widehat{\alpha}(\mu\vec{x}.\widehat{\alpha}(\vec{x}))} \rightleftharpoons \mathcal{WT}_{\mu\vec{x}.\widehat{\alpha}(\vec{x})}$ from Part 4 of Lemma 5.26. Fourth, take a narrow tableau $\mathcal{T}_{\mu\vec{x}.\widehat{\alpha}(\vec{x})}$, then, we have $\mathcal{WT}_{\mu\vec{x}.\widehat{\alpha}(\vec{x})} \rightharpoonup \mathcal{T}_{\mu\vec{x}.\widehat{\alpha}(\vec{x})}$, again from Part 3 of Lemma 5.26. Fifth, the equivalence $\mathcal{T}_{\mu\vec{x}.\widehat{\alpha}(\vec{x})} \rightleftharpoons \mathcal{T}_{\widehat{\varphi}}$ is trivial by the definition of $\widehat{\varphi}$. Finally, by applying Part 1 and 2 of Lemma 5.26 repeatedly, we obtain $\mathcal{T}_{\widehat{\alpha}(\widehat{\varphi})} \rightharpoonup \mathcal{T}_{\widehat{\varphi}}$. ☐

6 Completeness

In this section, we prove the completeness of Koz. In Subsection 6.1, we give the concept of *refutation* and show that every unsatisfiable formula has a refutation. We also introduce the concept of *thin refutation* and exhibit Claim (f). In Subsection 6.2, we prove the completeness of Koz by proving Claim (h) and (d), in that order.

6.1 Refutation

Definition 6.1 (Refutation). A well-named formula φ is given. *Refutation rules* for φ are defined as the rules of tableau, but this time, we modify the set of rules by adding an explicit weakening rule:

$$\frac{\Gamma}{\alpha, \Gamma}\ (\mathsf{Weak})$$

and, instead of the $(\triangledown)$-rule, we take the following $(\triangledown_r)$-rule:

$$\frac{\{\psi_k\} \cup \{\bigvee \Psi_n \mid n \in N_{\psi_k}\}}{\triangledown\Psi_1, \ldots, \triangledown\Psi_i, l_1, \ldots, l_j}\ (\triangledown_r)$$

where in the $(\vee_w)$-rule, we have $1 \leq k \leq i$, $\psi_k \in \Psi_k$, $N_{\psi_k} = \{n \in \omega \mid 1 \leq n \leq i,\ n \neq k\}$ and $l_1, \ldots, l_j \in \mathsf{Lit}(\varphi)$. Therefore the $(\triangledown_r)$-rule has one premise.

A *refutation* for φ is a structure $\mathcal{R}_\varphi = (T, C, r, L)$ where (T, C, r) is a tree structure and $L : T \to \mathcal{P}(\mathsf{Sub}(\varphi))$ is a *label function* satisfying the following clauses:

1. $L(r) = \{\varphi\}$.
2. Every leaf is labeled by some inconsistent set of formulas.
3. Let $t \in T$. If $L(t)$ is modal and inconsistent, then t has no child. Otherwise, if t is labeled by the set of formulas which fulfils the form of the conclusion of some refutation rules, then t has children which are labeled by the sets of formulas of premises of those refutation rules.
4. The rule $(\triangledown_r)$ can be applied to t only if $L(t)$ is modal.
5. For any infinite branch π, π is *odd* (not even) in the sense of Definition 5.6.

Lemma 6.2. *Let φ be a well-named formula. If φ is not satisfiable, then there exists a refutation for φ.*

Proof. From Lemmas 5.9 and 5.10, we find that φ is not satisfiable if and only if Player 3 has the memoryless winning strategy f_3 for the tableau game $\mathcal{TG}(\varphi)$. If Player 3 has the memoryless the winning strategy f_3, then winning tree $\mathcal{T}_\varphi | f_3$ derived by f_3 is a refutation for φ. ☐

Definition 6.3 (Aconjunctive formula). Let φ be a well-named formula, and $\preceq_\varphi$ be its dependency order (recall Definition 2.4). Then,

- For any $\psi \in \mathsf{Sub}(\varphi)$ and $x \in \mathsf{Bound}(\varphi)$, we say x is *active* in ψ if there exists $y \in \mathsf{Sub}(\psi) \cap \mathsf{Bound}(\varphi)$ such that $x \preceq_\varphi y$.

- A variable $x \in \mathsf{Bound}(\varphi)$ is called *aconjunctive* if, for any $\alpha \wedge \beta \in \mathsf{Sub}(\varphi_x(x))$, x is active in at most one of α or β.

- φ is called *aconjunctive* if every $x \in \mathsf{Bound}(\varphi)$ such that $\sigma_x = \mu$ is aconjunctive.

Definition 6.4 (Thin refutation). Let $\mathcal{R}_\varphi$ be a refutation for some well-named formula φ. We say that $\mathcal{R}_\varphi$ is *thin* if, whenever a formula of the form $\alpha \wedge \beta$ is reduced, some node of the refutation and some variable is active in α as well as β, then at least one of α and β is immediately discarded by using the (Weak)-rule.

From Definition 6.4 and Lemma 6.2, it is obvious that every unsatisfiable aconjunctive formula has a thin refutation (without the (Weak)-rule). The following Theorem 6.5 was first proved in Kozen [8] for the refutation of an aconjunctive formula, and then extended in the following way in Walukiewicz [15]. We will omit its proof.

Theorem 6.5. *Let φ be a well-named formula. If there exists a thin refutation for φ, then $\sim\varphi$ is probable in* Koz*.*

Corollary 6.6. *Let $\widehat{\varphi}$ be an automaton normal form. Then, we have*

1. *$\widehat{\varphi}$ is aconjunctive.*

2. *If $\widehat{\varphi}$ is not satisfiable, then $\vdash\sim\widehat{\varphi}$.*

Proof. The first assertion of the Corollary is obvious from the observation of Remark 5.13. For the second assertion, suppose that $\widehat{\varphi}$ is not satisfiable. Then, from Lemma 6.2, there exists a refutation for $\widehat{\varphi}$. Since $\widehat{\varphi}$ is aconjunctive, this refutation is thin and, thus, we have $\vdash\sim\widehat{\varphi}$ from Theorem 6.5. □

In the next Lemma, we confirm that some compositions preserve aconjunctiveness.

Lemma 6.7 (Composition). *Let φ, ψ and $\alpha(x)$ be aconjunctive formulas where $x \in$ Prop appears only positively in $\alpha(x)$. Then $\varphi \wedge \psi$, $\alpha(\varphi)$ and $\nu\vec{x}.\alpha(\vec{x})$ are also aconjunctive.*

Proof. We only prove the claim concerning $\alpha(\varphi)$ and the other two claims are left as exercises for the reader. As mentioned in Remark 2.11, $\alpha(\varphi)$ is an abbreviation of $\alpha(\varphi_1, \ldots, \varphi_k)$ where φ_i with $1 \le i \le k$ are appropriate renaming formulas of φ. For our purpose, the following assertions are fundamental;

$$\mathsf{Bound}(\alpha(x)) \cap \mathsf{Bound}(\varphi_i) = \emptyset \quad (1 \le \forall i \le k) \tag{13}$$

$$\mathsf{Bound}(\alpha(x)) \cap \mathsf{Free}(\varphi_i) = \emptyset \quad (1 \le \forall i \le k) \tag{14}$$

$$\mathsf{Bound}(\varphi_i) \cap \mathsf{Bound}(\varphi_j) = \emptyset \quad (1 \le i, j \le k, i \ne j) \tag{15}$$

Let $y \in \mathsf{Bound}(\alpha(\varphi))$ be a variable such that $\sigma_y = \mu$. From (13) and (15), we have $y \in \mathsf{Bound}(\alpha(x))$ or $y \in \mathsf{Bound}(\varphi_i)$ for some $i \in \omega$ such that $1 \le i \le k$. If $y \in \mathsf{Bound}(\alpha(x))$, then from (14), for every $z \in \mathsf{Bound}(\alpha(\varphi))$ such that $y \preceq_{\alpha(\varphi)} z$, we have $z \in \mathsf{Bound}(\alpha(x))$. Hence, y is aconjunctive in $\alpha(\varphi)$ if and only if y is aconjunctive in $\alpha(x)$. By a similar argument, we can show that if $y \in \mathsf{Bound}(\varphi_i)$, then y is aconjunctive in $\alpha(\varphi)$ if and only if y is aconjunctive in φ_i. From the above argument and the assumptions of the Lemma, we can assume that every bound variable y is aconjunctive in $\alpha(\varphi)$ and thus $\alpha(\varphi)$ is indeed aconjunctive. □

6.2 Proof of completeness

Lemma 6.8. *Let α be an aconjunctive formula, and $\widehat{\varphi}$ be an automaton normal form. A tableau $\mathcal{T}_\alpha = (T_\alpha, C_\alpha, r_\alpha, L_\alpha)$ for α and a tableau $\mathcal{T}_{\widehat{\varphi}} = (T_{\widehat{\varphi}}, C_{\widehat{\varphi}}, r_{\widehat{\varphi}}, L_{\widehat{\varphi}})$ for $\widehat{\varphi}$ are given. If $\mathcal{T}_{\widehat{\varphi}}$ is a tableau consequence of $\mathcal{T}_\alpha$, then we can construct a thin refutation $\mathcal{R}$ for $\alpha\wedge\sim\widehat{\varphi}$ ($\equiv \sim(\alpha \to \widehat{\varphi})$).*

Proof. Let $\mathcal{T}_\alpha$ and $\mathcal{T}_{\widehat{\varphi}}$ be the tableaux satisfying the condition of the Lemma. Then, there exists a tableau consequence relation Z from $\mathcal{T}_\alpha$ to $\mathcal{T}_{\widehat{\varphi}}$. Now, we will construct a thin refutation $\mathcal{R} = (T, C, r, L)$ for $\alpha\wedge\sim\widehat{\varphi}$ inductively. To facilitate the construction, we define two correspondence functions $\mathsf{Cor}_\alpha : T \to T_\alpha$ and

$\mathsf{Cor}_{\widehat{\varphi}} : T \to T_{\widehat{\varphi}}$. These functions are partial and, in every considered node t of $\mathcal{R}$, the following conditions are satisfied:

$$L(t) = L_\alpha(\mathsf{Cor}_\alpha(t)) \cup \left\{\sim \bigvee (L_{\widehat{\varphi}}(\mathsf{Cor}_{\widehat{\varphi}}(t)) \setminus \mathsf{Top})\right\} \tag{16}$$

$$(\mathsf{Cor}_\alpha(t), \mathsf{Cor}_{\widehat{\varphi}}(t)) \in Z \tag{17}$$

Of course, the root of $\mathcal{R}$ is labeled by $\{\alpha \wedge \sim\widehat{\varphi}\}$ and its child, say t_0, is labeled by $\{\alpha, \sim\widehat{\varphi}\}$. For the base step, set $\mathsf{Cor}_\alpha(t_0) := r_\alpha$ and $\mathsf{Cor}_{\widehat{\varphi}}(t_0) := r_{\widehat{\varphi}}$. Then, the Condition (16) and (17) are indeed satisfied. The remaining construction is divided into two cases; the second of which will be further divided into four cases.

Inductive step I Suppose we have already constructed $\mathcal{R}$ up to a node t where $\mathsf{Cor}_\alpha(t)$ and $\mathsf{Cor}_{\widehat{\varphi}}(t)$ are choice nodes of appropriate tableaux and satisfy Conditions (16) and (17). In this case, we prolong $\mathcal{R}$ up to u so that:

1. $\mathsf{Cor}_\alpha(u)$ is a modal node of $\mathcal{T}_\alpha$ near $\mathsf{Cor}_\alpha(t)$.
2. $\mathsf{Cor}_{\widehat{\varphi}}(u)$ is a modal node of $\mathcal{T}_{\widehat{\varphi}}$ near $\mathsf{Cor}_{\widehat{\varphi}}(t)$.
3. Conditions (16) and (17) are satisfied in u.
4. $\mathsf{TR}[t, u] \equiv \mathsf{TR}[\mathsf{Cor}_\alpha(t), \mathsf{Cor}_\alpha(u)] \cup \{\langle \sim\bigvee(L_{\widehat{\varphi}}(t_1) \setminus \mathsf{Top}), \cdots, \sim\bigvee(L_{\widehat{\varphi}}(t_k) \setminus \mathsf{Top})\rangle\}$ where $t_1 \cdots t_k \in T^+_{\widehat{\varphi}}$ is the $C_{\widehat{\varphi}}$-sequence starting at $\mathsf{Cor}_{\widehat{\varphi}}(t)$ and ending at $\mathsf{Cor}_{\widehat{\varphi}}(u)$.

The idea of the prolonging procedure is represented in Figure 9. From t, we first apply the tableau

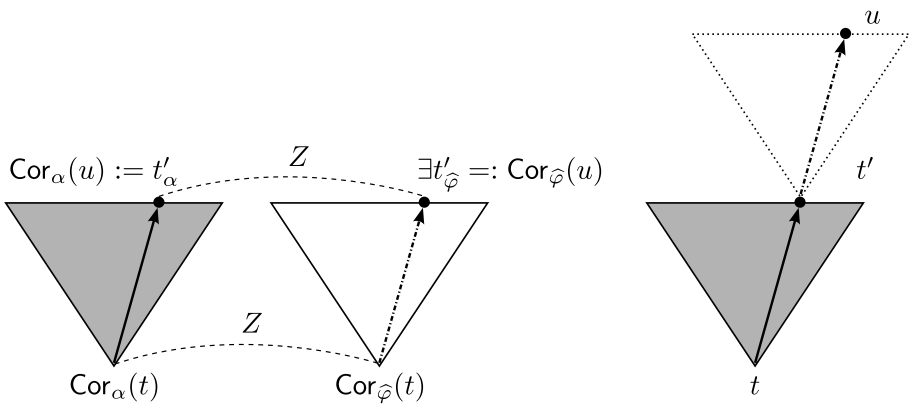

Figure 9: The prolonging procedure for Inductive step I.

rules to the formulas of $\mathsf{Sub}(L_\alpha(\mathsf{Cor}_\alpha(t)))$ in the same order as they were applied from $\mathsf{Cor}_\alpha(t)$ and its nearest modal nodes. Then, we obtain a finite tree rooted in t which is isomorphic to the section of $\mathcal{T}_\alpha$ between $\mathsf{Cor}_\alpha(t)$ and its nearest modal nodes. Therefore, for each leaf t' of this section of $\mathcal{R}$, we can take unique modal node t'_α of $\mathcal{T}_\alpha$ that is isomorphic to t'. Note that $L(t') = L_\alpha(t'_\alpha) \cup \{\sim\bigvee(L_{\widehat{\varphi}}(\mathsf{Cor}_{\widehat{\varphi}}(t)) \setminus \mathsf{Top})\}$. Now, the forth condition on the choice node of Z is used. From (17), we can find $t'_{\widehat{\varphi}} \in T_{\widehat{\varphi}}$ which is near $\mathsf{Cor}_{\widehat{\varphi}}(t)$ and satisfies $(t'_\alpha, t'_{\widehat{\varphi}}) \in Z$. Let us look at the path from $\mathsf{Cor}_{\widehat{\varphi}}(t)$ to $t'_{\widehat{\varphi}}$ in $\mathcal{T}_{\widehat{\varphi}}$. Since $\widehat{\varphi}$ is an automaton normal form on this path only the $(\vee)$-, (σ)- and (Reg)-rules, and $(\wedge)$-rules reducing $\widehat{\psi} \wedge \top_i$ to $\{\widehat{\psi}, \top_i\}$ may be applied first. Then, we have zero or more applications of the $(\wedge)$-rule. Let us apply dual rules to $\sim\bigvee L_{\widehat{\varphi}}(\mathsf{Cor}_{\widehat{\varphi}}(t))$ (note that (Reg) and (σ) are self-dual).

For an application of the $(\vee)$-rule in $\mathcal{T}_{\widehat{\varphi}}$, we apply the $(\wedge)$-rule followed by the (Weak)-rule to leave only the conjunct which appears on the path to $t'_{\widehat{\varphi}}$. In this way, we ensure the resulting path of $\mathcal{R}$ will be thin.

For an application of the $(\wedge)$-rule reducing $\widehat{\psi} \wedge \top_i$ to $\{\widehat{\psi}, \top_i\}$ in $\mathcal{T}_{\widehat{\varphi}}$, we apply the $(\vee)$-rule in $\mathcal{R}$. Then, we have two children, say v_1 and v_2 such that $L(v_1)$ includes $\sim\widehat{\psi}$ and $L(v_2)$ includes $\sim\top_i = \bot$. Since $L(v_2)$ is inconsistent, if we further prolong $\mathcal{R}$ from v_2 to its nearest modal nodes, such modal nodes also labeled inconsistent set. This means that the modal nodes can be leaves of a refutation. We therefore stop the prolonging procedure on such modal nodes.

After these reductions, we get a node u which is labeled by $L_\alpha(t'_\alpha) \cup \{\sim \bigvee(L_{\widehat{\varphi}}(t'_{\widehat{\varphi}}) \setminus \mathsf{Top})\}$. Setting $\mathsf{Cor}_\alpha(u) := t'_\alpha$ and $\mathsf{Cor}_{\widehat{\varphi}}(u) := t'_{\widehat{\varphi}}$ establishes Conditions (16) and (17). Conditions 1 through 4 follow directly from the construction.

Inductive step II Suppose we have already constructed $\mathcal{R}$ up to a node t where $\mathsf{Cor}_\alpha(t)$ and $\mathsf{Cor}_{\widehat{\varphi}}(t)$ are modal nodes of appropriate tableaux and satisfy Conditions (16) and (17). Note that, since $\widehat{\varphi}$ is an automaton normal form, we can put $L_{\widehat{\varphi}}(\mathsf{Cor}_{\widehat{\varphi}}(t)) \setminus \mathsf{Top} = \{\nabla\Psi, l_1, \ldots, l_i\}$ or $L_{\widehat{\varphi}}(\mathsf{Cor}_{\widehat{\varphi}}(t)) \setminus \mathsf{Top} = \{l_1, \ldots, l_i\}$ where $l_1, \ldots, l_i \in \mathsf{Lit}(\widehat{\varphi})$. Moreover, observe that

$$\begin{aligned}
\sim\left(\nabla\Psi \wedge \bigwedge_{1\le k\le i} l_k\right) &\equiv \sim\nabla\Psi \vee \left(\bigvee_{1\le k\le i} \sim l_k\right) \\
&\equiv \sim\Big(\Big(\bigwedge \Diamond\Psi\Big) \wedge \Box\Big(\bigvee \Psi\Big)\Big) \vee \left(\bigvee_{1\le k\le i} \sim l_k\right) \\
&\equiv \left(\bigvee_{\psi\in\Psi} \Box\sim\psi\right) \vee \Diamond\Big(\bigwedge \sim\Psi\Big) \vee \left(\bigvee_{1\le k\le i} \sim l_k\right) \\
&\equiv \left(\bigvee_{\psi\in\Psi} (\nabla\{\sim\psi\} \vee \nabla\emptyset)\right) \vee \nabla\Big\{\Big(\bigwedge\sim\Psi\Big), \top\Big\} \vee \left(\bigvee_{1\le k\le i} \sim l_k\right).
\end{aligned}$$

Therefore, if we prolong $\mathcal{R}$ from t up to its nearest modal nodes u by applying the $(\vee)$-rule repeatedly, the label of u can be categorized as one of following four cases:

(Case 1): $L(u) = L_\alpha(\mathsf{Cor}_\alpha(t)) \cup \{\sim l_k\}$ for some k such that $1 \le k \le i$.

(Case 2): $L(u) = L_\alpha(\mathsf{Cor}_\alpha(t)) \cup \{\nabla\emptyset\}$.

(Case 3): $L(u) = L_\alpha(\mathsf{Cor}_\alpha(t)) \cup \{\nabla\{\sim\psi\}\}$ for some $\psi \in \Psi$.

(Case 4): $L(u) = L_\alpha(\mathsf{Cor}_\alpha(t)) \cup \{\nabla\{(\bigwedge\sim\Psi), \top\}\}$.

In every cases, it is possible that $L_\alpha(\mathsf{Cor}_\alpha(t))$ is inconsistent and, thus, $L(u)$ is also inconsistent. If this is so, all u can be a leaf of a refutation. Therefore, we stop the prolonging procedure on u in this case. Now, we consider the case where $L_\alpha(\mathsf{Cor}_\alpha(t))$ is consistent.

In Case 1, the prop condition is used; by Condition (17), we have $l_k \in L_\alpha(\mathsf{Cor}_\alpha(t))$. Thus, $L(u)$ includes l_k and $\sim l_k$. This means that $L(u)$ is inconsistent and so u can be a leaf of a refutation. We therefore stop the prolonging procedure on u in this case.

In Case 2, the back condition on modal nodes is used. Since $C_{\widehat{\varphi}}(\mathsf{Cor}_{\widehat{\varphi}}(t)) \ne \emptyset$, it must hold that $C_\alpha(\mathsf{Cor}_\alpha(t)) \ne \emptyset$. Take $v_\alpha \in C_\alpha(\mathsf{Cor}_\alpha(t))$ arbitrarily. We prolong $\mathcal{R}$ from u to $v \in C(u)$ in such a way that $L(v) = L_\alpha(v_\alpha) \cup \{\bigvee\emptyset(\equiv \bot)\}$. Since $L(v)$ is inconsistent, if we further prolong $\mathcal{R}$ from v to its nearest modal nodes, such modal nodes are also inconsistent. This means that the modal nodes can be a leaves of a refutation. We therefore stop the prolonging procedure on such modal nodes in this case.

In Case 3, the back condition on modal nodes is used. Let $v_{\widehat{\varphi}}$ be a child of $\mathsf{Cor}_{\widehat{\varphi}}(t)$ such that $L_{\widehat{\varphi}}(v_{\widehat{\varphi}}) = \{\psi\}$. Then, by Condition (17), we can find $v_\alpha \in C_\alpha(\mathsf{Cor}_\alpha(t))$ such that $(v_\alpha, v_{\widehat{\varphi}}) \in Z$. We create a new child v of u which is labeled by $L_\alpha(\mathsf{Cor}_\alpha(v_\alpha)) \cup \{\sim\psi\}$. Moreover, we set $\mathsf{Cor}_\alpha(v) := v_\alpha$ and $\mathsf{Cor}_{\widehat{\varphi}}(v) := v_{\widehat{\varphi}}$. This prolonging procedure preserves Conditions (16) and (17). Note that, in this case, $\mathsf{Cor}_\alpha(v)$ and $\mathsf{Cor}_{\widehat{\varphi}}(v)$ are choice nodes of appropriate tableaux.

In Case 4, the forth condition on modal nodes is used. The idea of the prolonging procedure is represented in Figure 10. Let $L_\alpha(\mathsf{Cor}_\alpha(t)) = \{\nabla\Delta_1, \ldots, \nabla\Delta_i, l_1, \ldots, l_j\}$. In this case, we first create a new child v of u such that

$$L(v) = \Big\{\bigvee\Delta_1, \ldots, \bigvee\Delta_i\Big\} \cup \Big\{\bigwedge\sim\Psi\Big\}.$$

From the choice node v, we further prolong $\mathcal{R}$ up to its nearest modal nodes t' so that

5. $\mathsf{Cor}_\alpha(t')$ is a next modal node of $\mathsf{Cor}_\alpha(t)$.

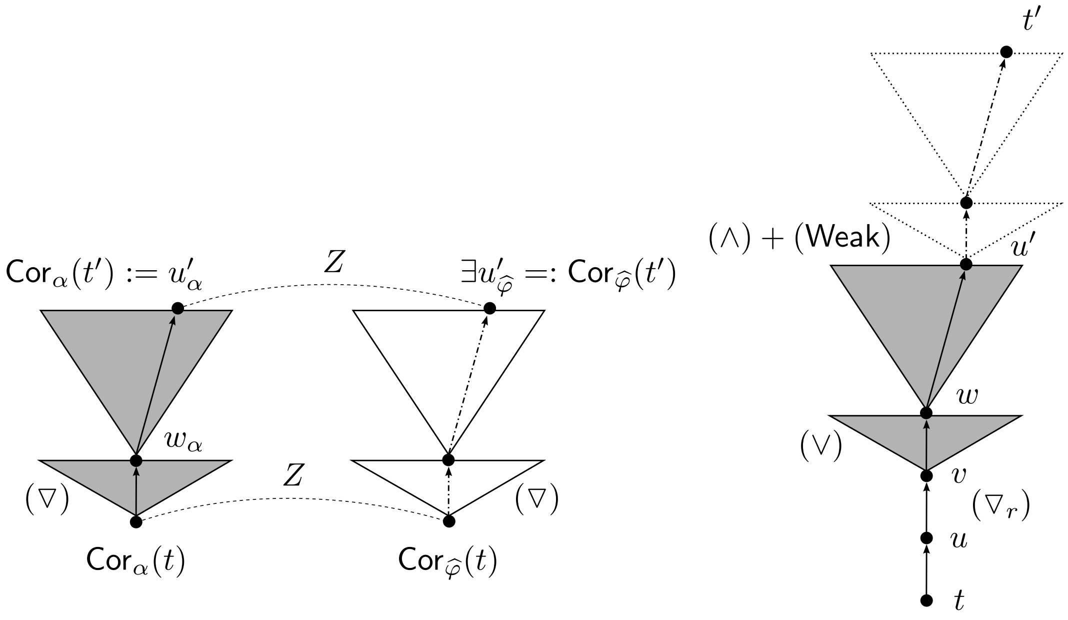

Figure 10: The prolonging procedure for Case 4.

6. $\mathsf{Cor}_{\widehat{\varphi}}(t')$ is a next modal node of $\mathsf{Cor}_{\widehat{\varphi}}(t)$.
7. Condition (16) and (17) are satisfied in t'.
8. $\mathsf{TR}[u, t'] \equiv \mathsf{TR}^+[\mathsf{Cor}_\alpha(t), \mathsf{Cor}_\alpha(t')] \cup \{\langle \triangledown \{(\bigwedge \sim \Psi), \top\}, \bigwedge \sim \Psi, \ldots, \sim\psi = \sim \bigvee (L_{\widehat{\varphi}}(t_1) \setminus \mathsf{Top}), \cdots, \sim \bigvee (L_{\widehat{\varphi}}(t_k) \setminus \mathsf{Top})\rangle\}$ where $t_1 \cdots t_k \in T^+_{\widehat{\varphi}}$ is the $C_{\widehat{\varphi}}$-sequence starting at the child of $\mathsf{Cor}_{\widehat{\varphi}}(t)$ labeled by $\{\psi\}$ and ending at $\mathsf{Cor}_{\widehat{\varphi}}(t')$.

Next, we apply $(\vee)$-rules to $\bigvee \Delta_1$ repeatedly until we arrive at the node w such that

$$L(w) = \{\delta_1\} \cup \left\{\bigvee \Delta_2, \ldots, \bigvee \Delta_i\right\} \cup \left\{\bigwedge \sim \Psi\right\}$$

where $\delta_1 \in \Delta_1$. Note that there exists $w_\alpha \in C_\alpha(\mathsf{Cor}_\alpha(t))$ such that

$$L_\alpha(w_\alpha) = \{\delta_1\} \cup \left\{\bigvee \Delta_2, \ldots, \bigvee \Delta_i\right\}$$

From w, we apply the tableau rules to formulas of $\mathsf{Sub}(L_\alpha(w_\alpha))$ in the same order as they were applied from w_α and its nearest modal nodes. Then, we obtain a finite tree rooted in w which is isomorphic to the section of $\mathcal{T}_\alpha$ between w_α and nearest modal nodes. Therefore, for each leaf u' of this section of $\mathcal{R}$, we can take a unique modal node u'_α of $\mathcal{T}_\alpha$ which is isomorphic to u'. Note that $L(u') = L_\alpha(u'_\alpha) \cup \{\bigwedge \sim \Psi\}$. Since u'_α is a next modal node of $\mathsf{Cor}_\alpha(t)$, from Condition (17) and the forth condition on modal nodes, we can assume that there exists $u'_{\widehat{\varphi}}$ which is a next modal node of $\mathsf{Cor}_{\widehat{\varphi}}(t)$ and satisfies $(u'_\alpha, u'_{\widehat{\varphi}}) \in Z$. We will now look at the path from $\mathsf{Cor}_{\widehat{\varphi}}(t)$ to $t'_{\widehat{\varphi}}$ in $\mathcal{T}_{\widehat{\varphi}}$ and exploit $(\wedge)$-rules and (Weak)-rules so that the trace tr on this path satisfies Condition 8. Finally, we get a node t' which is labeled by $L_\alpha(u'_\alpha) \cup \{\sim \bigvee L_{\widehat{\varphi}}(u'_{\widehat{\varphi}})\}$. Setting $\mathsf{Cor}_\alpha(t') := u'_\alpha$ and $\mathsf{Cor}_{\widehat{\varphi}}(t') := u'_{\widehat{\varphi}}$ establishes Conditions (16) and (17). Then, Conditions 5 through 8 follow directly from the construction.

The above two procedures completely describe $\mathcal{R}$. All the leaves are labeled by an inconsistent set. Moreover, take an infinite branch π of $\mathcal{R}$ arbitrarily. Let π_α be the branch of $\mathcal{T}_\alpha$ such that $\{n \in \omega \mid \mathsf{Cor}_\alpha(\pi) = \pi_\alpha[n]\}$ is an infinite set. Let $\pi_{\widehat{\varphi}}$ be the branch of $\mathcal{T}_{\widehat{\varphi}}$ such that $\{n \in \omega \mid \mathsf{Cor}_{\widehat{\varphi}}(\pi) = \pi_{\widehat{\varphi}}[n]\}$ is an infinite set. For any trace $\mathsf{tr} \in \mathsf{TR}(\pi)$, we have $\mathsf{tr}[1] = \alpha \wedge \sim \widehat{\varphi}$ and, $\mathsf{tr}[2] = \alpha$ or $\mathsf{tr}[2] = \sim \widehat{\varphi}$. $\mathsf{TR}_1(\pi)$ denotes the set of all the trace $\mathsf{tr} \in \mathsf{TR}(\pi)$ such that $\mathsf{tr}[2] = \alpha$. $\mathsf{tr}_2 \in \mathsf{TR}(\pi)$ denotes the trace such that $\mathsf{tr}_2[2] = \sim \widehat{\varphi}$. Then, from the construction of $\mathcal{R}$, we have;

(T1) $\mathsf{TR}(\pi) = \mathsf{TR}_1(\pi) \cup \{\mathsf{tr}_2\}$.

(T2) $\mathsf{TR}_1^+(\pi) \equiv \mathsf{TR}^+(\pi_\alpha)$.

(T3) tr_2 is even if and only if $\pi_{\widehat{\varphi}}$ is odd.

(T4) π_α and $\pi_{\widehat{\varphi}}$ are associated with each other.

Above conditions imply that π is odd. Indeed, if π_α is odd, then, from **(T2)**, π is also odd. If π_α is even, then, from **(T4)**, $\pi_{\widehat{\varphi}}$ is also even. Therefore, from **(T3)**, tr_2 is odd. From **(T1)**, we can assume that π is odd. $\mathcal{R}$ is also thin because α is aconjunctive and whenever we reduce a $\wedge$-formula originated from $\sim\widehat{\varphi}$, we leave only one conjunction and discard the other by applying (Weak)-rule. Therefore, $\mathcal{R}$ is a thin refutation as required. ☐

Lemma 6.9 (Main lemma). *For any well-named formula φ, there exists a semantically equivalent automaton normal form $\widehat{\varphi}$ such that $\varphi \to \widehat{\varphi}$ is provable in* Koz*. Moreover, for any $x \in \mathsf{Free}(\varphi)$ which occurs only positively in φ, it hold that $x \in \mathsf{Free}(\widehat{\varphi})$ and x occurs only positively in $\widehat{\varphi}$.*

Proof. We prove the lemma by the induction on the structure of φ.

Case: $\varphi \in \mathsf{Lit}$**.** In this case, $\widehat{\varphi}$ is just φ.

Case: $\varphi = \alpha \vee \beta$**.** By the induction assumption, there exist automaton normal forms $\widehat{\alpha}$ and $\widehat{\beta}$ which are equivalent to α and β, respectively, such that $\vdash \alpha \to \widehat{\alpha}$ and $\vdash \beta \to \widehat{\beta}$. Set $\widehat{\varphi} := \widehat{\alpha} \vee \widehat{\beta}$. Then, we have $\vdash \alpha \vee \beta \to \widehat{\varphi}$.

Case: $\varphi = \nabla\Psi$**.** This case is very similar to the previous one.

Case: $\varphi = \alpha \wedge \beta$**.** By the induction assumption, there exist automaton normal forms $\widehat{\alpha}$ and $\widehat{\beta}$ which are equivalent to α and β respectively, such that $\vdash \alpha \to \widehat{\alpha}$ and $\vdash \beta \to \widehat{\beta}$; thus, we have $\vdash \alpha \wedge \beta \to \widehat{\alpha} \wedge \widehat{\beta}$. Set $\widehat{\varphi} := \mathsf{anf}(\widehat{\alpha} \wedge \widehat{\beta})$. Then, from Theorem 5.17, we have $\mathcal{T}_{\widehat{\alpha}\wedge\widehat{\beta}} \rightleftharpoons \mathcal{T}_{\widehat{\varphi}}$ and, thus, $\mathcal{T}_{\widehat{\alpha}\wedge\widehat{\beta}} \rightharpoonup \mathcal{T}_{\widehat{\varphi}}$. On the other hand, by Lemma 6.7, we can assume that $\widehat{\alpha} \wedge \widehat{\beta}$ is aconjunctive. From Lemma 6.8 and Theorem 6.5, we have $\vdash \widehat{\alpha} \wedge \widehat{\beta} \to \widehat{\varphi}$. Therefore, we have $\vdash \alpha \wedge \beta \to \widehat{\varphi}$.

Case: $\varphi = \nu x_1. \dots \nu x_k.\alpha(x_1, \dots, x_k)$**.** By the induction assumption, we have an equivalent automaton normal form $\widehat{\alpha}(x)$ of $\alpha(x)$ such that $\vdash \alpha(x) \to \widehat{\alpha}(x)$. Therefore, $\vdash \nu\vec{x}.\alpha(\vec{x}) \to \nu\vec{x}.\widehat{\alpha}(\vec{x})$. Set $\widehat{\varphi} := \mathsf{anf}(\nu\vec{x}.\widehat{\alpha}(\vec{x}))$. Then, from Theorem 5.17, we have $\mathcal{T}_{\nu\vec{x}.\widehat{\alpha}(\vec{x})} \rightleftharpoons \mathcal{T}_{\widehat{\varphi}}$ and, thus, $\mathcal{T}_{\nu\vec{x}.\widehat{\alpha}(\vec{x})} \rightharpoonup \mathcal{T}_{\widehat{\varphi}}$. On the other hand, by Lemma 6.7, we can assume that $\nu\vec{x}.\widehat{\alpha}(\vec{x})$ is aconjunctive. From Lemma 6.8 and Theorem 6.5, we have $\vdash \nu\vec{x}.\widehat{\alpha}(\vec{x}) \to \widehat{\varphi}$. Therefore, $\vdash \nu\vec{x}.\alpha(\vec{x}) \to \widehat{\varphi}$.

Case: $\varphi = \mu x_1. \dots \mu x_k.\alpha(x_1, \dots, x_k)$**.** By the induction assumption, we have an equivalent automaton normal form $\widehat{\alpha}(x)$ of $\alpha(x)$ such that $\vdash \alpha(x) \to \widehat{\alpha}(x)$. Therefore, $\vdash \mu\vec{x}.\alpha(\vec{x}) \to \mu\vec{x}.\widehat{\alpha}(\vec{x})$. Set $\widehat{\varphi} := \mathsf{anf}(\mu\vec{x}.\widehat{\alpha}(\vec{x}))$. Then, from Corollary 5.27, we have $\mathcal{T}_{\widehat{\alpha}(\widehat{\varphi})} \rightharpoonup \mathcal{T}_{\widehat{\varphi}}$. On the other hand, by Lemma 6.7, we can assume that $\widehat{\alpha}(\widehat{\varphi})$ is aconjunctive. From Lemma 6.8 and Theorem 6.5, $\vdash \widehat{\alpha}(\widehat{\varphi}) \to \widehat{\varphi}$. By applying the (Ind)-rule, we obtain $\vdash \mu\vec{x}.\widehat{\alpha}(\vec{x}) \to \widehat{\varphi}$. Thus, $\vdash \mu\vec{x}.\alpha(\vec{x}) \to \widehat{\varphi}$.

Hence, we have proved the Lemma for all cases. ☐

Theorem 6.10 (Completeness). *For any formula φ, if φ is not satisfiable, then $\sim\varphi$ is provable in* Koz*.*

Proof. Let φ be an unsatisfiable formula. By Part 5 of Lemma 2.9, we can construct a well-named formula $\mathsf{wnf}(\varphi)$ such that

$$\vdash \varphi \leftrightarrow \mathsf{wnf}(\varphi) \tag{18}$$

On the other hand, from Lemma 6.9, there exists an automaton normal form $(\mathsf{wnf}(\varphi))\hat{}$ which is semantically equivalent to $\mathsf{wnf}(\varphi)$ and thus to φ such that

$$\vdash \mathsf{wnf}(\varphi) \to (\mathsf{wnf}(\varphi))\hat{} \tag{19}$$

Since $(\mathsf{wnf}(\varphi))\hat{}$ is not satisfiable, by Corollary 6.6 we have

$$\vdash (\mathsf{wnf}(\varphi))\hat{} \to \bot \tag{20}$$

Finally by combining Equations (18) through (20) we obtain $\vdash \varphi \to \bot$ as required. ☐

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