Hintikka set

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A Hintikka set Γ is a set of FOL formulas fulfilling the following properties:

1. For each formula φ , either $\varphi \notin \Gamma$ or $\neg\varphi \notin \Gamma$.
2. $\varphi \to \psi \in \Gamma$ implies that either $\neg\varphi \in \Gamma$ or $\psi \in \Gamma$.
3. $\neg\neg\varphi \in \Gamma$ implies that $\varphi \in \Gamma$.
4. $\neg(\varphi \to \psi) \in \Gamma$ implies that $\varphi \in \Gamma$ and $\neg\psi \in \Gamma$.
5. $\forall x\varphi \in \Gamma$ implies that $\varphi(x/t) \in \Gamma$ for each term t of the language.
6. $\neg\forall x\varphi \in \Gamma$ implies that there is a term t of the language such that $\neg\varphi(x/t) \in \Gamma$.

Meaning of the new notation:

- In points 5 and 6, $\varphi(x/t) \in \Gamma$ means that each occurrence of y in φ' has been replaced by t, where φ' is the outcome of the α-renaming of $\forall x\varphi$ to $\forall y\varphi'$ with a fresh variable y, that is, $\varphi' = S^x_y\varphi$.

Theorem 1. *A Hintikka set Γ is satisfiable, i.e, there is some interpretation $\mathscr{I}$ and some $\sigma \in \Sigma_{\mathscr{I}}$ such that $(\mathscr{I}, \sigma) \Vdash \varphi$ for each $\varphi \in \Gamma$.*

Proof. We construct a Tarski structure $\mathscr{I} = \langle \mathcal{D}, \mathcal{I} \rangle$ and an assignment $\sigma \in \Sigma_{\mathscr{I}}$ such that $(\mathscr{I}, \sigma) \Vdash \varphi$ for each $\varphi \in \Gamma$.

- The domain $\mathcal{D}$ is defined as $\{\, t \mid t \text{ is a term of the language} \,\}$.
- For each $f \in FS$, $\mathcal{I}(f)(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$.
- For each $P \in PS$, $\mathcal{I}(P)(t_1, \ldots, t_n) = 1$ iff $P(t_1, \ldots, t_n) \in \Gamma$.
- For each $x \in VS$, $\sigma(x) = x$.

It is immediate to see that for each constant symbol a, $\mathcal{I}(a) = a$ and, by a trivial induction proof, that for each term t of the language, $\mathscr{I}(t)(\sigma) = t$.

By induction on the structure of the formula φ, we prove a stronger property:

$$\varphi \in \Gamma \implies (\mathscr{I}, \sigma) \Vdash \varphi$$
$$\neg\varphi \in \Gamma \implies (\mathscr{I}, \sigma) \nVdash \varphi$$

- If $\varphi = P(t_1, \ldots, t_n)$ and $\varphi \in \Gamma$, then

$$\begin{aligned}\mathscr{I}(\varphi)(\sigma) &= \mathcal{I}(P)(\mathscr{I}(t_1)(\sigma), \ldots, \mathscr{I}(t_n)(\sigma)) \\ &= \mathcal{I}(P)(t_1, \ldots, t_n) \\ &= 1,\end{aligned}$$

  thus $(\mathscr{I}, \sigma) \Vdash \varphi$

- If $\varphi = P(t_1, \ldots, t_n)$ and $\neg\varphi \in \Gamma$, then by condition 1 of the Hintikka set we have that $\varphi \notin \Gamma$, thus $\mathscr{I}(\varphi)(\sigma) = \mathcal{I}(P)(t_1, \ldots, t_n) = 0$, hence $(\mathscr{I}, \sigma) \nVdash \varphi$

- If $\varphi = \neg\psi$ and $\varphi \in \Gamma$, then we have that $\neg\psi \in \Gamma$, thus by condition 1 of the Hintikka set we have that $\psi \notin \Gamma$, hence by induction hypothesis $(\mathscr{I}, \sigma) \nVdash \psi$, thus $(\mathscr{I}, \sigma) \Vdash \neg\psi$

- If $\varphi = \neg\psi$ and $\neg\varphi \in \Gamma$, by condition 3 of the Hintikka set it follows tthat $\psi \in \Gamma$. By induction hypothesis we have $(\mathscr{I}, \sigma) \Vdash \psi$, thus $(\mathscr{I}, \sigma) \nVdash \neg\psi$

- If $\varphi = \psi \to \eta$ and $\varphi \in \Gamma$, implies by condition 2 of the Hintikka set that either $\neg\psi \in \Gamma$ or $\eta \in \Gamma$. By induction hypothesis we have $(\mathscr{I}, \sigma) \nVdash \psi$ or $(\mathscr{I}, \sigma) \Vdash \eta$, thus $(\mathscr{I}, \sigma) \Vdash \varphi$

- If $\varphi = \psi \to \eta$ and $\neg\varphi \in \Gamma$, implies by condition 4 of the Hintikka set that $\psi \in \Gamma$ and $\neg\eta \in \Gamma$. By induction hypothesis we have $(\mathscr{I}, \sigma) \Vdash \psi$ and $(\mathscr{I}, \sigma) \nVdash \eta$, thus $(\mathscr{I}, \sigma) \nVdash \varphi$

- If $\varphi = \forall x\psi$ and $\varphi \in \Gamma$, then by condition 5 of the Hintikka set it follows for each for each term t of the language, $\psi(x/t) \in \Gamma$. By induction hypothesis, for each term t of the language, $(\mathscr{I}, \sigma) \Vdash \psi(x/t)$, that is, $\mathscr{I}(\psi(x/t))(\sigma) = 1$. Since $\mathcal{D} = \{\, t \mid t \text{ is a term of the language} \,\}$ it follows that for each $t \in \mathcal{D}$, $\mathscr{I}(\psi)(\sigma[x/t]) = 1$, thus $(\mathscr{I}, \sigma) \Vdash \varphi$

- If $\varphi = \forall x\psi$ and $\neg\varphi \in \Gamma$, then for some t, $\neg\psi(x/t) \in \Gamma$, hence by condition 1 of the Hintikka set it follows that $\psi(x/t) \notin \Gamma$. By induction hypothesis, $(\mathscr{I}, \sigma) \nVdash \psi(x/t)$, hence $(\mathscr{I}, \sigma) \nVdash \varphi$

Now we have completed all cases of the induction proof, hence the given Hintikka set Γ is satisfiable. $\square$

Remark 2. *Because of the α-renaming, one may use arbitrary many fresh variables. Since only countable situations are concerned, the assignment σ is well-defined.*