

Exercise sheet 2 on Discrete Mathematics

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Let A be a finite alphabet, such as the English alphabet, and consider the following simple programming language **WHILE** whose well-formed programs are those obtained by applying the rules of the following grammar:

Numbers:

$$Num ::= d \mid dNum \quad \text{where } d \in \{0, 1, \dots, 9\}$$

Identifiers:

$$\begin{aligned} Id &::= aId' && \text{where } a \in A \\ Id' &::= \lambda \mid aId' \mid NumId' \end{aligned}$$

Numeric expressions:

$$\begin{aligned} Exp &::= Num \mid Id \\ &\mid Exp + Exp \mid Exp - Exp \mid Exp * Exp \mid Exp / Exp \end{aligned}$$

Boolean expressions:

$$\begin{aligned} BExp &::= \mathbf{true} \mid \mathbf{false} \mid Exp = Exp \\ &\mid \neg BExp \mid BExp \wedge BExp \mid BExp \vee BExp \end{aligned}$$

Programs:

$$\begin{aligned} P &::= \mathbf{skip} \mid Id := Exp \mid P; P \\ &\mid \mathbf{if } BExp \mathbf{ then } P \mathbf{ else } P \mathbf{ fi} \\ &\mid \mathbf{while } BExp \mathbf{ do } P \mathbf{ done} \end{aligned}$$

Exercise 2.1. Show, by providing the appropriate functions, that the set of identifiers has the same cardinality as \mathbb{N} .

Hint. Make use of results of the Exercise 1.4.2 in the lecture notes.

Exercise 2.2. Show that the set of all well-formed **WHILE** programs is countable.

Exercise 2.3. Show the following statements:

1. If A is an uncountable set and B is a countable set, then $|A \setminus B| = |A|$.
2. Let \mathbb{I} be the set of irrational numbers, i.e., $\mathbb{I} = \{ r \in \mathbb{R} \mid r \notin \mathbb{Q} \}$. Then $|\{0, 1\}^\omega \setminus \{0, 1\}^*0^\omega| = |\mathbb{I} \cap [0, 1]|$.
3. $|\{0, 1\}^\omega| = |[0, 1]|$.
4. $|[0, 1]| = |[0, 1] \times [0, 1]| = |[0, 1]^n|$ for each $n \geq 1$.

Hint. For point 3, make use of points 1 and 2; for point 4, make use of point 3.

Exercise 2.4. Consider the semantic bracket operator presented in Definition 2.1.6 of the lecture notes. Show that $\models \varphi \rightarrow \psi$ iff $\llbracket \varphi \rrbracket \subseteq \llbracket \psi \rrbracket$.

Exercise 2.5. Let $\Omega_1 \subseteq FOF$ be the set of FOL formulas φ such that each quantifier operator Qx (with $Q \in \{\forall, \exists\}$ and $x \in VS$) appears at most once in φ .

Provide a function *Scope*, including its type, such that, given a quantifier operator Qx (with $Q \in \{\forall, \exists\}$ and $x \in VS$) and a formula $\varphi \in \Omega_1$, it returns the formula corresponding to the scope of Qx .

Exercise 2.6. Given $\varphi, \psi, \eta \in FOF$,

1. provide a function $\#$, including its type, that returns how many times ψ occurs in φ as sub-formula;
2. provide a function R_∞ , including its type, that replaces each occurrence of φ in η with ψ ;
3. provide a function R_1 , including its type, that replaces exactly one occurrence of φ in η with ψ if φ occurs in η , and that returns η if φ does not occur in η . If φ occurs in η multiple times, then there is no requirement on the particular instance to be replaced.

Exercise 2.7. Prove the following:

1. For any predicate P with arity 2, $\forall x \forall y P(x, y) \vdash \forall y \forall z P(y, z)$.
2. Assume x is not free in φ , then $\varphi \rightarrow \forall x \psi$ and $\forall x(\varphi \rightarrow \psi)$ are syntactically equivalent.
3. We say a formula φ has repeated occurrences of a bound variable x , if Qx appears more than once in the sub-formulas of φ (recall $Q \in \{\forall, \exists\}$). Prove that there exists a formula φ' which has no repeated occurrences of any bound variable such that φ and φ' are syntactically equivalent.