# Exercise sheet 1 on Discrete Mathematics

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We first define the following normal forms.

**Conjunctive Normal Form (CNF):**

- Every formula $\varphi$ is a conjunction of clauses $\mathcal{C}_1, \ldots, \mathcal{C}_n$.
- A clause $\mathcal{C}$ is the disjunction of literals $\mathtt{l}_1, \ldots, \mathtt{l}_m$.
- A literal $\mathtt{l}$ is an atomic proposition or the negation of an atomic proposition.

$$\begin{aligned} \varphi &::= \mathcal{C} \mid \mathcal{C} \wedge \varphi \\ \mathcal{C} &::= \mathtt{l} \mid \mathtt{l} \vee \mathcal{C} \\ \mathtt{l} &::= p \mid \neg p \qquad\qquad \text{(where } p \in AP\text{)} \end{aligned}$$

**Disjunctive Normal Form (DNF):**

- Every formula $\varphi$ is a disjunction of clauses $\mathcal{C}_1, \ldots, \mathcal{C}_n$.
- A clause $\mathcal{C}$ is the conjunction of literals $\mathtt{l}_1, \ldots, \mathtt{l}_m$.
- A literal $\mathtt{l}$ is an atomic proposition or the negation of an atomic proposition.

$$\begin{aligned} \varphi &::= \mathcal{C} \mid \mathcal{C} \vee \varphi \\ \mathcal{C} &::= \mathtt{l} \mid \mathtt{l} \wedge \mathcal{C} \\ \mathtt{l} &::= p \mid \neg p \qquad\qquad \text{(where } p \in AP\text{)} \end{aligned}$$

**Negation-free Normal Form (NNF):**

- Negation may appear only in front of atomic propositions.

$$\varphi ::= p \mid \neg p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \qquad\qquad \text{(where } p \in AP\text{)}$$

**Exercise 1.1** (Relations between normal forms)**.** *Prove formally the following statements (cf. Exercises 43–45 on page 35).*

- *For each BNF formula (as defined in Definition 1.1.2)* $\varphi_b$ *there is a NNF formula* $\varphi_n$ *such that* $\varphi_b \equiv \varphi_n$ *and vice-versa.*
- *For each CNF formula* $\varphi_c$ *there is a DNF formula* $\varphi_d$ *such that* $\varphi_c \equiv \varphi_d$ *and vice-versa.*
- *For each BNF formula (as defined in Definition 1.1.2)* $\varphi_b$ *there is a CNF formula* $\varphi_c$ *such that* $\varphi_b \equiv \varphi_c$ *and vice-versa.*

**Exercise 1.2.** *Explain which rules have been used in the proofs for all examples from Example 1.5.7 to Example 1.5.17.*

**Exercise 1.3.** *Show, by applying the rules of the deduction system presented in Section 1.5, the following statements:*

1. $\vdash (\varphi \to \psi) \to ((\neg\varphi \to \neg\psi) \to (\psi \to \varphi))$
2. $\vdash ((\varphi \to (\psi \to \eta)) \to (\varphi \to \psi)) \to ((\varphi \to (\psi \to \eta)) \to (\varphi \to \eta))$
3. $\vdash (\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$
4. $\vdash \varphi \to (\psi \to (\varphi \to \psi))$
5. $\{\varphi \to \psi, \neg(\psi \to \eta) \to \neg\varphi\} \vdash \varphi \to \eta$
6. $\varphi \to (\psi \to \eta) \vdash \psi \to (\varphi \to \eta)$
7. $\vdash ((\varphi \to \psi) \to \varphi) \to \varphi$
8. $\vdash \neg(\varphi \to \psi) \to (\psi \to \varphi)$

**Exercise 1.4.** *Find a deduction showing the correctness of some of the following equivalences, that is, if* $\varphi \equiv \psi$*, then provide a deduction for* $\vdash \varphi \to \psi$ *and for* $\vdash \psi \to \varphi$*.*

1. $\varphi \vee (\varphi \wedge \psi) \equiv \varphi$,
2. $(\varphi_1 \to \varphi_2) \vee (\varphi_1 \to \varphi_3) \equiv \varphi_1 \to (\varphi_2 \vee \varphi_3)$.

**Exercise 1.5** (* not required)**.** *Fill the missing parts of the proofs of the soundness and completeness theorems in Section 1.5.*