Discrete Mathematics[1]
http://iscasmc.ios.ac.cn/DM2016

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[1]Materials about the axiomatization are provided by Dr. Wanwei Liu.

Contents

Chapter 4

Algorithms

4.1 Syntax of an algorithm in pseudo-code

In this part, we consider an extension of the programming language **WHILE** presented in the Exercise sheet 2. Basically, the grammar is extended with *arrays* and *procedures*.

Let A be a finite alphabet, such as the English alphabet, and consider the following simple programming language **PROC**, an extension of **WHILE**, whose well-formed programs are those obtained by applying the rules of the following grammar:

Numbers:

$$\mathit{Num} ::= d \mid d\mathit{Num} \qquad\qquad \text{where } d \in \{0, 1, \ldots, 9\}$$

Identifiers:

$$\begin{aligned} \mathit{Id} &::= a\mathit{Id}' && \text{where } a \in A \\ \mathit{Id}' &::= \lambda \mid a\mathit{Id}' \mid \mathit{Num}\mathit{Id}' \end{aligned}$$

Numeric expressions:

$$\begin{aligned} \mathit{Exp} ::= {} & \mathit{Num} \mid \mathit{Id} \mid \mathit{Id}[\mathit{Exp}] \mid \mathbf{length}(\mathit{Id}) \mid (\mathit{Exp}) \\ & \mid \mathit{Exp} + \mathit{Exp} \mid \mathit{Exp} - \mathit{Exp} \mid \mathit{Exp} * \mathit{Exp} \mid \mathit{Exp}/\mathit{Exp} \\ & \mid \mathit{Id}(\mathit{Exp}, \ldots, \mathit{Exp}) \end{aligned}$$

Boolean expressions:

$$\begin{aligned} \mathit{BExp} ::= {} & \mathbf{true} \mid \mathbf{false} \mid \mathit{Exp} = \mathit{Exp} \mid (\mathit{BExp}) \\ & \mid \mathit{Exp} \leq \mathit{Exp} \mid \mathit{Exp} < \mathit{Exp} \\ & \mid \mathit{Exp} \geq \mathit{Exp} \mid \mathit{Exp} > \mathit{Exp} \\ & \mid \neg \mathit{BExp} \mid \mathit{BExp} \land \mathit{BExp} \mid \mathit{BExp} \lor \mathit{BExp} \end{aligned}$$

Procedure declaration

$$\begin{aligned} \mathit{Pr} ::= {} & \ \mathbf{procedure}\ \mathit{Id}(\mathit{Id}, \ldots, \mathit{Id})\ P \\ & \mid \mathbf{procedure}\ \mathit{Id}(\mathit{Id}, \ldots, \mathit{Id})\ P;\ \mathbf{return}\ \mathit{Exp} \end{aligned}$$

Programs:

$$\begin{aligned} P ::= {} & \mathbf{skip} \mid \mathit{Id} := \mathit{Exp} \mid \mathit{Id}[\mathit{Exp}] := \mathit{Exp} \mid P; P \\ & \mid \mathbf{if}\ \mathit{BExp}\ \mathbf{then}\ P\ \mathbf{else}\ P\ \mathbf{fi} \\ & \mid \mathbf{while}\ \mathit{BExp}\ \mathbf{do}\ P\ \mathbf{done} \\ & \mid \mathbf{for}\ \mathit{Id} := \mathit{Exp}\ \mathbf{to}\ \mathit{Exp}\ \mathbf{do}\ P\ \mathbf{done} \\ & \mid \mathit{Id}(\mathit{Exp}, \ldots, \mathit{Exp}) \end{aligned}$$

4.2 Examples of algorithms

Algorithm 4.2.1 (Maximum in a generic array)**.** The following pseudo-code in **PROC** returns the maximum value occurring in the provided `array`.

```
procedure max(array)
  maxvalue := array[0];
  for i := 1 to length(array) − 1 do
    if array[i] > maxvalue then
      maxvalue := array[i]
    else
      skip
    fi
  done;
  return maxvalue
```

Algorithm 4.2.2 (Index of a value in an array)**.** The following pseudo-code in **PROC** returns the index of the specified `value` if it occurs in the provided `array`, otherwise **length**(`array`) is returned.

```
procedure indexOf(value, array)
  index := length(array);
  for i := 0 to length(array) − 1 do
    if array[i] = value then
      index := i
    else
      skip
    fi
  done;
  return index
```

Algorithm 4.2.3 (Index of a value in a sorted array)**.** The following pseudo-code in **PROC** returns the index of the specified `value` if it occurs in the provided sorted `array`, otherwise **length**(`array`) is returned.

```
procedure indexOfSorted(value, array)
  index := length(array);
  low := 0;
  high := length(array) − 1;
  while low < high do
    middle := (low + high)/2;
    if array[middle] < value then
      low := middle + 1
    else
      high := middle
    fi
  done;
  if array[low] = value then
    index := low
  else
    skip
  fi;
  return index
```

Algorithm 4.2.4 (Swap elements in an array)**.** The following pseudo-code in **PROC** swaps the elements at index `i` and `j` in the provided `array`.

```
procedure swap(array, i, j)
  temp := array[i];
  array[i] := array[j];
  array[j] := temp
```

Algorithm 4.2.5 (Bubble sort)**.** The following pseudo-code in **PROC** sorts the provided `array`.

```
procedure bubbleSort(array)
  for i := 0 to length(array) − 1 do
    for j := 0 to (length(array) − 1) − i do
      if array[j] > array[j + 1] then
        swap(array, j, j + 1)
      else
        skip
      fi
    done
  done
```

Algorithm 4.2.6 (Gnome sort)**.** The following pseudo-code in **PROC** sorts the provided `array`.

```
procedure gnomeSort(array)
  i := 0;
  while i < length(array) do
    if i = 0 ∨ array[i − 1] ≤ array[i] then
      i := i + 1
    else
      swap(array, i, i − 1);
      i := i − 1
    fi
  done
```

Algorithm 4.2.7 (Insertion sort)**.** The following pseudo-code in **PROC** sorts the provided `array`.

```
procedure insertionSort(array)
  for i := 1 to length(array) − 1 do
    j := i;
    while j > 0 ∧ array[j − 1] > array[j] do
      swap(array, j, j − 1);
      j := j − 1
    done
  done
```

Greedy Algorithms

Algorithm 4.2.8 (Change Making)**.** The algorithm makes changes $c_1 > c_2 \ldots > c_r$ for n cents.

```
procedure procedureChange($c_1, c_2, \ldots, c_r$)
  for $i := 1$ to $r$ do
    $d_i := 0$;
    while $n \geq c_i$ do
      $d_i := d_i + 1$;
      $n := n - c_i$
    done
  done
```

Algorithm 4.2.9 (Earliest Ending Time Job Scheduling)**.** Two jobs are *compatible* if they do not overlap. Assumption: sorts talks with ending times $e_1 < e_2 < \ldots e_n$.

```
procedure schedule($e_1, e_2, \ldots, e_n$)
  $S := \emptyset$;
  for $i := 1$ to $n$ do
    if talk $i$ is compatible with $S$ then $S := S \cup \{talk\ i\}$
  done
```

Remark: set not defined in **PROC**

Definition 4.2.10. *The* Tableau *approach for the satisfiability problem of propositional logic: a propositional formula is satisfiable iff it has a consistent tableau.*

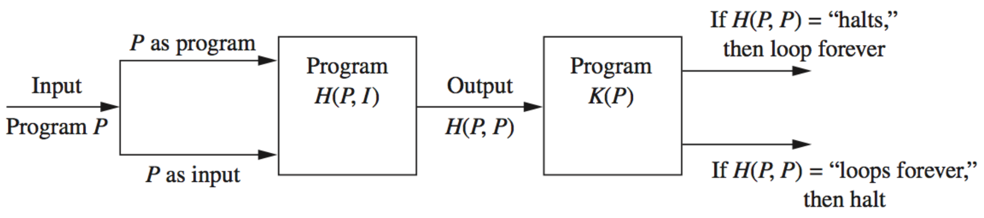
Definition 4.2.11 (Decidability)**.**

- *A decision problem consists of a set of instances and a subset of yes-instances.*
- *The Primality problem: is the instance x a prime number?*
- *The answer (solution) to any decision problem is just one bit (true or false).*
- *A problem Q is* decidable *iff there is an algorithm A, such that for each instance q of Q, the computation $A(q)$ stops with an answer.*
- *A problem Q is semi-decidable iff there is an algorithm A, such that for each instance q of Q, if q holds, then $A(q)$ stops with the positive answer; otherwise, $A(q)$ either stops with the negative answer,* or does not stop.

Definition 4.2.12 (The Halting Problem)**.** *It takes as input a computer program and input to the program and determines whether the program will eventually stop when run with this input.*

- *If the program halts, we have our answer.*
- *If it is still running after any fixed length of time has elapsed, we do not know whether it will never halt or we just did not wait long enough for it to terminate.*

Definition 4.2.13 (Undecidability of the Halting Problem)**.**


Input
Program P
P as program
P as input
Program H(P, I)
Output
H(P, P)
Program K(P)
If H(P, P) = "halts," then loop forever
If H(P, P) = "loops forever," then halt

4.3 The Growth of Functions

Definition 4.3.1. *Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $f(x)$ is $O(g(x))$ if there are constants C and k such that $|f(x)| \leq C|g(x)|$ whenever $x > k$.*

Definition 4.3.2. *Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $f(x)$ is $\Omega(g(x))$ if there are positive constants C and k such that $|f(x)| \geq C|g(x)|$ whenever $x > k$.*

Moreover, We say that $f(x)$ is $\Theta(g(x))$ if $f(x)$ is $O(g(x))$ and $\Omega(g(x))$.

Exercise 4.3.3. *Show that:*

1. *Show that log $n!$ is $O(nlogn)$.*
2. *Show that n^2 is not $O(n)$.*
3. *Suppose that $f_1(x)$ is $O(g_1(x))$ and that $f_2(x)$ is $O(g_2(x))$. Then $(f_1 + f_2)(x)$ is $O(max(|g_1(x)|, |g_2(x)|))$.*
4. *Suppose that $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$. Then $(f_1 f_2)(x)$ is $O(g_1(x)g_2(x))$.*

5. *Show that* $3x^2 + 8x \log x$ *is* $\Theta(x^2)$.

6. *Assume* $a_n \neq 0$. *Show that* $\sum_{i=0}^{n} a_i x^i$ *is* $\Theta(x^n)$.

4.4 Complexity of Algorithms

We are interested in the *time complexity* of the algorithm.

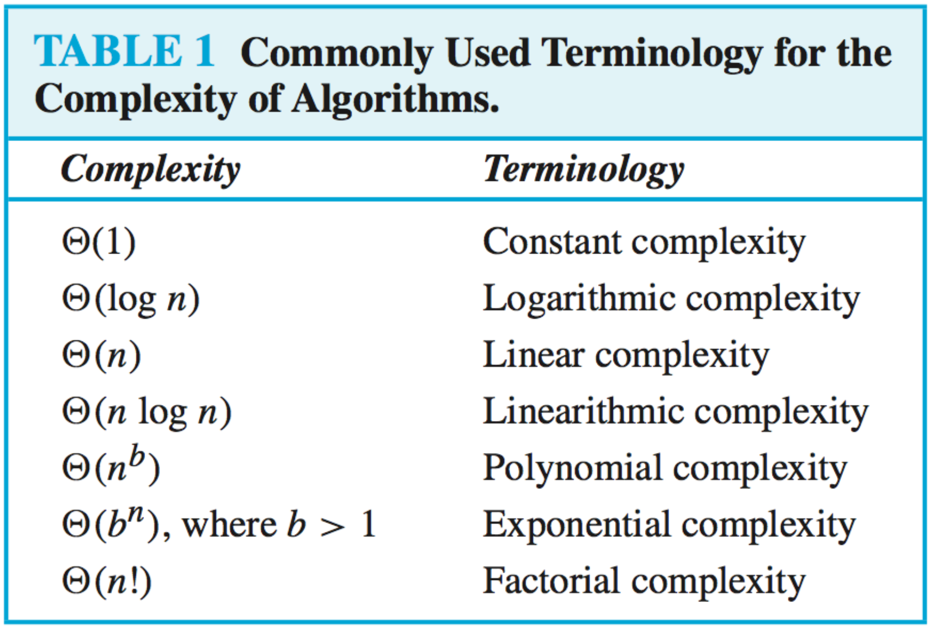


TABLE 1 Commonly Used Terminology for the Complexity of Algorithms.

Complexity	*Terminology*
$\Theta(1)$	Constant complexity
$\Theta(\log n)$	Logarithmic complexity
$\Theta(n)$	Linear complexity
$\Theta(n \log n)$	Linearithmic complexity
$\Theta(n^b)$	Polynomial complexity
$\Theta(b^n)$, where $b > 1$	Exponential complexity
$\Theta(n!)$	Factorial complexity

Definition 4.4.1.

- *A problem that is solvable using an algorithm with polynomial worst-case complexity is called tractable. Tractable problems are said to belong to class* P.
- *Problems for which a solution can be checked in polynomial time are said to belong to the class* NP.
- *The* $P = NP$ *problem asks whether* NP, *the class of problems for which it is possible to check solutions in polynomial time, equals* P, *the class of tractable problems.*

- *NP-complete problems: It is an NP problem and if a polynomial time algorithm for solving it were known, then $P = NP$.*
- *The satisfiability problem is NP-complete. Cook-Levin Theorem*

A prize of $1,000,000$ *dollars is offered by the Clay Mathematics Institute for its solution.*

4.5 Logic and Computer Science – Logical Revolution

The following material is adapted from the slides by Vardi.

Definition 4.5.1 (Hilbert's Program)**.** *Hilbert's Program (1922-1930): Formalize mathematics and establish that:*

- *Mathematics is consistent:*

  a mathematical statement and its negation cannot ever both be proved.

- *Mathematics is complete:*

  all true mathematical statements can be proved.

- *Mathematics is decidable: there is a mechanical way to determine whether a given mathematical statement is true or false.*

Definition 4.5.2 (The Demise of Hilbert's Program)**.** *Gödel:*

- *Incompleteness of ordinary arithmetic - There is no systematic way of resolving all mathematical questions.*

- *Impossibility of proving consistency of mathematics*

  Gödel (1930): "This sentence is not provable."

- *Church and Turing (1936): Unsolvability of first-order logic:*

  The set of valid first-order sentences is not computable.

Definition 4.5.3 (Entscheidungsproblem)**.** *Entscheidungsproblem (The Decision Problem) [Hilbert-Ackermann, 1928]: Decide if a given first-order sentence is valid (dually, satisfiable).*

Church-Turing Theorem, 1936: The Decision Problem is unsolvable.

Turing, 1936:

- *Defined computability in terms of Turing machines (TMs)*
- *Proved that the termination problem for TMs is unsolvable ("this machine terminates iff it does not terminate")*
- *Reduced termination to Entscheidungsproblem.*

Definition 4.5.4 (Mathematical Logic - 1936)**.** *Logic as Foundations of Mathematics:*

- *Incomplete (example: Continuum Hypothesis)*
- *Cannot prove its own consistency*
- *Unsolvable decision problem*
- *Unsolvable termination problem*

  Can we get some positive results?

- *Focus on special cases!*

Definition 4.5.5 (The Fragment-Classification Project)**.** *Idea: Identify decidable fragments of first-order logic - (1915-1983)*

- *Monadic Class (monadic predicates)*
- *Bernays-Schönfinkel Class ($\exists^*\forall^*$)*
- *Ackermann Class ($\exists^*\forall\exists^*$)*
- *Gödel Class ($\exists^*\forall\forall\exists^*$)*

Outcome: Very weak classes! What good is first-order logic?

Definition 4.5.6 (Monadic Logic)**.** *Monadic Class: First-order logic with monadic predicates - captures syllogisms.*
$\forall x(Man(x) \to Mortal(x))$
Löwenheim, 1915: The Monadic Class is decidable.

- *Proof: Bounded-model property - if a sentence is satisfiable, it is satisfiable in a structure of bounded size.*
- *Proof technique: quantifier elimination.*

Definition 4.5.7 (Logic of Integer Addition)**.** *Integer Addition:*

- *Domain: N (natural numbers)*
- *Predicate: $=$*
- *Addition function: $+$*
- $y = 2x : y = x + x$
- $x \leq y : (\exists z)(y = x + z)$
- $x = 0 : (\forall y)(x \leq y)$
- $x = 1 : x \neq 0 \wedge (\forall y)(y = 0 \vee x \leq y)$
- $y = x + 1 : (\exists z)(z = 1 \wedge y = x + z)$

Bottom Line: Theory of Integer Addition can express Integer Programming (integer inequalities) and much more.

Definition 4.5.8 (Presburger Arithmetics)**.** *Mojzesz Presburger, 1929:*

- *Sound and complete axiomatization of integer addition*
- *Decidability: There exists an algorithm that decides whether a given first-order sentence in integer-addition theory is true or false.*
  - *Decidability is shown using quantifier elimination, supplemented by reasoning about arithmetical congruences.*
  - *Decidability can also be shown using automata-theoretic techniques.*

Definition 4.5.9 (Complexity of Presburger Arithmetics)**.** *Complexity Bounds:*

- *Oppen, 1978: TIME($2^{2^{2^{poly}}}$) upper bound*
- *Fischer & Rabin, 1974: TIME($2^{2^{lin}}$) lower bound*

Rabin, 1974: "Theoretical Impediments to Artificial Intelligence": "the complexity results point to a need for a careful examination of the goals and methods in AI".

Definition 4.5.10 (Finite Words - Nondeterministic Finite Automata)**.** $A = (\Sigma, S, S_0, \rho, F)$

- *Alphabet:* Σ
- *States:* S
- *Initial states:* $S_0 \subseteq S$

- *Nondeterministic transition function:* $\rho \colon S \times \Sigma \to 2^S$
- *Accepting states:* $F \subseteq S$

Definition 4.5.11. *Finite Words - Nondeterministic Finite Automata*

Input word: $a_0, a_1, ..., a_{n1}$
Run: $s_0, s_1, ..., s_n$

- $s_0 \in S_0$
- $s_{i+1} \in \rho(s_i, a_i)$ *for* $i \geq 0$

Acceptance: $s_n \in F$
Recognition: $L(A)$ *- words accepted by* A*.*
Fact: NFAs define the class Reg of regular languages.

Definition 4.5.12 (Logic of Finite Words)**.** *View finite word* $w = a_0, ..., a_{n1}$ *over alphabet* Σ *as a mathematical structure:*

- *Domain:* $0, ..., n-1$
- *Dyadic predicate:* $\leq$
- *Monadic predicates:* $\{P_a : a \in \Sigma\}$

Monadic Second-Order Logic (MSO):

- *Monadic atomic formulas:* $P_a(x)$ *(*$a \in \Sigma$*)*
- *Dyadic atomic formulas:* $x < y$
- *Set quantifiers:* $\exists P, \forall P$
  Example: $(\exists x)((\forall y)(\neg(x < y)) \wedge P_1(x))$ *- last letter is* 1*.*

Definition 4.5.13 (Automata and Logic)**.** *[Büchi, Elgot, Trakhtenbrot, 1957-8 (independently)]:* MSO $\equiv$ NFA

Both MSO and NFA define the class Reg.

Proof: Effective

- *From NFA to MSO* ($A \to \varphi_A$)
- *From MSO to NFA* ($\varphi \to A_\varphi$)

Definition 4.5.14 (NFA Nonemptiness)**.** *Nonemptiness:* $L(A) = \emptyset$

Nonemptiness Problem: Decide if given A, $L(A)$ *is nonempty.*

Directed Graph $G_A = (S, E)$ *of NFA* $A = (\Sigma, S, S_0, \rho, F)$*:*

- *Nodes:* S
- *Edges:* $E = \{(s,t) : t \in \rho(s,a) \text{ for some } a \in \Sigma\}$

It holds: A *is nonempty iff there is a path in* G_A *from* S_0 *to* F.

Decidable in time linear in size of A*, using breadth-first search or depth-first search.*

Definition 4.5.15 (MSO Satisfiability - Finite Words)**.** *Satisfiability:* $models(\psi) = \emptyset$

Satisfiability Problem: Decide if given ψ *is satisfiable.*

It holds: ψ *is satisfiable iff* A_ψ *is nonnempty.*

It holds: MSO satisfiability is decidable.

- *Translate* ψ *to* A_ψ.
- *Check nonemptiness of* A_ψ .

Definition 4.5.16 (Complexity Barrier)**.** *Computational Complexity:*

- *Naive Upper Bound: Nonelementary Growth* 2 *to the power of the tower of height* $O(n)$

- *Lower Bound [Stockmeyer, 1974]: Satisfiability of FO over finite words is nonelementary (no bounded-height tower).*

Definition 4.5.17 (Program Verification)**.** *The Dream - Hoare, 1969: "When the correctness of a program, its compiler, and the hardware of the computer have all been established with mathematical certainty, it will be possible to place great reliance on the results of the program."*

The Nightmare - De Millo, Lipton, and Perlis, 1979: "We believe that... program verification is bound to fail. We cannot see how it is going to be able to affect anyone's confidence about programs."

The Hoare Triple $\{\varphi\}P\{\psi\}$

Definition 4.5.18 (Logic in Computer Science: c. 1980)**.** *Status: Logic in CS is not too useful!*

- *First-order logic is undecidable.*
- *The decidable fragments are either too weak or too intractable.*
- *Even Boolean logic is intractable.*
- *Program verification is hopeless.*

Definition 4.5.19. *Post 1980: From Irrelevance to Relevance*

A Logical Revolution:

- *Relational databases*
- *Boolean reasoning*

- *Model checking*
- *Termination checking*
- *...*

Definition 4.5.20 (The Temporal Logic of Programs)**.** *Crux: Need to specify ongoing behavior rather than input/output relation! "Temporal logic to the rescue" [Pnueli, 1977]:*

- *Linear temporal logic (LTL) as a logic for the specification of non-terminating programs*
- *Model checking via reduction to MSO*

  But: nonelementary complexity!

In 1996, Pnueli received the Turing Award for seminal work introducing temporal logic into computing science and for outstanding contributions to program and systems verification.

Definition 4.5.21 (Examples)**.**

- *always not (CS1 and CS2): safety*
- *always (Request implies eventually Grant): liveness*
- *always (Request implies (Request until Grant)): liveness*

Definition 4.5.22 (Model Checking)**.** *"Algorithmic verification" [Clarke & Emerson, 1981, Queille & Sifakis, 1982]: Model checking programs of size m wrt CTL formulas of size n can be done in time mn.*

Linear-Time Response [Lichtenstein & Pnueli, 1985]: Model checking programs of size m wrt LTL formulas of size n can be done in time $m2^{O(n)}$ (tableau heuristics).

Seemingly:

- *Automata: non-elementary*
- *Tableaux: exponential*

Definition 4.5.23 (Back to Automata)**.** *Exponential-Compilation Theorem [Vardi & Wolper, 1983-1986]: Given an LTL formula φ of size n, one can construct an automaton A_φ of size $2^{O(n)}$ such that a trace σ satisfies φ if and only if σ is accepted by A_φ. Automata-Theoretic Algorithms:*

- *LTL Satisfiability: φ is satisfiable iff $L(A_\varphi) = \emptyset$ (PSPACE)*
- *LTL Model Checking: $M \models \varphi$ iff $L(M \times A_{\neg\varphi}) = \emptyset$ ($m2^{O(n)}$)*

Today: Widespread industrial usage
Industrial Languages: PSL, SVA (IEEE standards)

Definition 4.5.24 (Solving the Unsolvable)**.** *B. Cook, A. Podelski, and A. Rybalchenko, 2011: "in contrast to popular belief, proving termination is not always impossible"*

- *The Terminator tool can prove termination or divergence of many Microsoft programs.*
- *Tool is not guaranteed to terminate! Explanation:*
- *Most real-life programs, if they terminate, do so for rather simple reasons.*
- *Programmers almost never conceive of very deep and sophisticated reasons for termination.*

Definition 4.5.25 (Logic: from failure to success)**.** *Key Lessons:*

- *Algorithms*
- *Heuristics*

- *Experimentation*
- *Tools and systems*

Key Insight: Do not be scared of worst-case complexity.

- *It barks, but it does not necessarily bite!*