Discrete Mathematics[1]
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Contents

Chapter 3

First Order Logic (FOL)

3.1 Syntax of FOL

Propositional logic is a coarse language, which only concerns about propositions and boolean connectives. Practically, this logic is not powerful enough to describe important properties we are interested in.

Example 3.1.1 (Syllogism of Aristotle)**.** *Consider the following assertions:*

1. *All men are mortal.*
2. *Socrates is a man.*
3. *So Socrates would die.*

$$\forall x(Man(x) \to Mortal(x))$$

Definition 3.1.2. *First order logic is an extension of proposition logic:*

1. *To accept parameters, it generalized propositions to predicates.*

2. *To designate elements in the domain, it is equipped with functions and constants.*

3. *It also involves quantifiers to capture infinite conjunction and disjunction.*

Definition 3.1.3. *We are given:*

- *an arbitrary set of variable symbols* $VS = \{x, y, x_1, \dots\}$*;*
- *an arbitrary set (maybe empty) of function symbols* $FS = \{f, g, f_1, \dots\}$*, where each symbol has an arity;*
- *an arbitrary set (maybe empty) of predicate symbols* $PS = \{P, Q, P_1, \dots\}$*, where each symbol has an arity;*
- *an equality symbol set* ES *which is either empty or one element set containing* $\{\approx\}$*.*

Let $L = VS \cup \{(,), \to, \neg, \forall\} \cup FS \cup PS \cup ES$*. Here* $VS \cup \{(,), \to, \neg, \forall\}$ *are referred to as* logical symbols*, and* $FS \cup PS \cup ES$ *are referred to as* non-logical symbols*.*

We often make use of the

- *set of constant symbols, denoted by* $CS = \{a, b, a_1, \dots\} \subseteq FS$*, which consist of function symbols with arity* 0*;*
- *set of propositional symbols, denoted by* $PS = \{p, q, p_1, \dots\} \subseteq FS$*, which consist of predicate symbols with arity* 0*.*

Definition 3.1.4 (FOL terms)**.** *The terms of the first order logic are constructed according to the following grammar:*

$$t ::= x \mid f t_1 \dots t_n$$

where $x \in VS$*, and* $f \in FS$ *has arity* n*.*

Accordingly, the set T *of terms is the smallest set satisfying the following conditions:*

- *each variable $x \in VS$ is a term.*
- *Compound terms: $ft_1 \dots t_n$ is a term (thus in T), provided that f is a n-arity function symbol, and $t_1, \dots, t_n \in T$. Particularly, $a \in CS$ is a term.*

We often write $f(t_1, \dots, t_n)$ for the compound terms.

Definition 3.1.5 (FOL formulas)**.** *The well-formed formulas of the first order logic are constructed according to the following grammar:*

$$\varphi ::= Pt_1 \dots t_n \mid \neg\varphi \mid \varphi \to \varphi \mid \forall x\varphi$$

where $t_1, \dots, t_n$ are terms, $P \in PS$ has arity n, and $x \in VS$.

We often write $P(t_1, \dots, t_n)$ for clarity. Accordingly, the set FOF of first order formulas is the smallest set satisfying:

- *$P(t_1, \dots, t_n) \in FOF$ is a formula, referred to as the atomic formula.*
- *Compound formulas: $(\neg\varphi)$ (negation), $(\varphi \to \psi)$ (implication), and $(\forall x\varphi)$ (universal quantification) are formulas (thus in FOF), provided that $\varphi, \psi \in FOF$.*

We omit parentheses if it is clear from the context.

As syntactic sugar, we can define $\exists x\varphi$ as $\exists x\varphi := \neg\forall x\neg\varphi$. We assume that $\forall$ and $\exists$ have higher precedence than all logical operators.

Definition 3.1.6 (Sub-formulas)**.** *For a formula φ, we define*

the sub-formula function $Sf : FOF \to 2^{FOF}$ *as follows:*

$$\begin{aligned} Sf(P(t_1, \ldots, t_n)) &= \{P(t_1, \ldots, t_n)\} \\ Sf(\neg\varphi) &= \{\neg\varphi\} \cup Sf(\varphi) \\ Sf(\varphi \to \psi) &= \{\varphi \to \psi\} \cup Sf(\varphi) \cup Sf(\psi) \\ Sf(\forall x\varphi) &= \{\forall x\varphi\} \cup Sf(\varphi) \\ Sf(\exists x\varphi) &= \{\exists x\varphi\} \cup Sf(\varphi) \end{aligned}$$

Definition 3.1.7 (Scope)**.** *The part of a logical expression to which a quantifier is applied is called the scope of this quantifier. Formally, each sub-formula of the form* $Qx\psi \in Sf(\varphi)$*, the scope of the corresponding quantifier* Qx *is* ψ*. Here* $Q \in \{\forall, \exists\}$*.*

Substitution for Terms

Definition 3.1.8 (Sentence)**.** *We say an occurrence of* x *in* φ *is free if it is not in scope of any quantifiers* $\forall x$ *(or* $\exists x$*). Otherwise, we say that this occurrence is a bound occurrence. If a variable* φ *has no free variables, it is called a* closed formula*, or a* sentence*.*

Definition 3.1.9 (Substitution)**.** *The substitution of* x *with* t *within* φ*, denoted as* $S_t^x\varphi$*, is obtained from* φ *by replacing each free occurrence of* x *with* t*.*

We would extend this notation to $S_{t_1,\ldots,t_n}^{x_1,\ldots,x_n}\varphi$.

Remark 3.1.10. *It is important to remark that* $S_{t_1,\ldots,t_n}^{x_1,\ldots,x_n}\varphi$ *is not the same as* $S_{t_1}^{x_1} \ldots S_{t_n}^{x_n}\varphi$*: the former performs a simultaneous substitution.*

For example, consider the formula $P(x, y)$*: the subsitution* $S_{y,x}^{x,y}P(x, y)$ *gives* $S_{y,x}^{x,y}P(x, y) = P(y, x)$ *while the substitutions* $S_y^x S_x^y P(x, y)$ *give* $S_y^x S_x^y P(x, y) = S_y^x P(x, x) = P(y, y)$*.*

Remark 3.1.11. *Consider* $\varphi = \exists y(x < y)$ *in the number theory. What is* $S_t^x\varphi$ *for the special case of* $t = y$*?*

Definition 3.1.12 (Substitutable on Terms)**.** *We say that* t *is substitutable for* x *within* φ *iff for each variable* y *occurring in* t*, there is no free occurrence of* x *in scope of* $\forall y/\exists y$ *in* φ*.*

Definition 3.1.13 (α-β condition)**.** *If the formula* φ *and the variables* x *and* y *fulfill:*

1. y *has no free occurrence in* φ*, and*
2. y *is substitutable for* x *within* φ*,*

then we say that φ*,* x *and* y *meet the* α-β *condition, denoted as* $C(\varphi, x, y)$*.*

Lemma 3.1.14. *If* $C(\varphi, x, y)$*, then* $S_x^y S_y^x \varphi = \varphi$*.*

3.2 The Axiom System: the Hilbert's System

As for propositional logic, also FOL can be axiomatized.

Definition 3.2.1 (Axioms)**.**

1. $\varphi \to (\psi \to \varphi)$
2. $(\varphi \to (\psi \to \eta)) \to ((\varphi \to \psi) \to (\varphi \to \eta))$
3. $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$
4. $\forall x\varphi \to S_t^x\varphi$
   if t *is substitutable for* x *within* φ
5. $\forall x(\varphi \to \psi) \to (\forall x\varphi \to \forall x\psi)$
6. $\varphi \to \forall x\varphi$
   if x *is not free in* φ

7. $\forall x_1 \ldots \forall x_n \varphi$

   if φ is an instance of (one of) the above axioms

MP Rule: $\dfrac{\varphi \to \psi \quad \varphi}{\psi}$

Definition 3.2.2 (Syntactical Equivalence)**.** *We say φ and ψ are syntactically equivalent iff $\varphi \vdash \psi$ and $\psi \vdash \varphi$.*

Theorem 3.2.3. *(Gen): If x has no free occurrence in Γ, then $\Gamma \vdash \varphi$ implies $\Gamma \vdash \forall x \varphi$.*

Solution. Suppose that $\varphi_0, \varphi_1, \ldots, \varphi_n = \varphi$ is the deductive sequence of φ from Γ.

- If φ_i is an instance of some axiom, then according to (AS7), $\forall x \varphi_i$ is also an axiom.
- If $\varphi_i \in \Gamma$, since x is not free in Γ, we have $\vdash \varphi_i \to \forall x \varphi_i$ according to (AS6). Therefore, we have $\Gamma \vdash \forall x \varphi_i$ in this case.
- If φ_i is obtained by applying (MP) to some φ_j and $\varphi_k = \varphi_j \to \varphi_i$. By induction, we have $\Gamma \vdash \forall x \varphi_j$ and $\Gamma \vdash \forall x(\varphi_j \to \varphi_i)$. With (AS5) and (MP), we also have $\Gamma \vdash \forall x \varphi_i$ in this case.

Thus, we have $\Gamma \vdash \forall x \varphi_n$, i.e., $\Gamma \vdash \forall x \varphi$.

Exercise 3.2.4. *Prove that*

1. $\forall x \forall y \varphi \vdash \forall y \forall x \varphi$,
2. $\exists x \forall y \varphi \vdash \forall y \exists x \varphi$.

Exercise 3.2.5. *Prove that*

1. $\forall x(\varphi \to \psi) \vdash \forall x(\neg\psi \to \neg\varphi)$,

2. $\forall x(\varphi \to \psi) \vdash \exists x\varphi \to \exists x\psi$.

Exercise 3.2.6. *Prove that*

1. *If* $\Gamma \vdash \varphi$ *and* $\Gamma \vdash \neg\psi$, *then* $\Gamma \vdash \neg(\varphi \to \psi)$,

2. $\forall x\neg(\varphi \to \psi) \vdash \neg(\varphi \to \exists x\psi)$.

Lemma 3.2.7. *(Ren): If* $C(\varphi, x, y)$, *then* $\forall x\varphi$ *and* $\forall y S^x_y\varphi$ *are syntactical equivalent. That is,*

1. $\forall x\varphi \vdash \forall y S^x_y\varphi$.

2. $\forall y S^x_y\varphi \vdash \forall x\varphi$.

Lemma 3.2.8. *(RS): Let* η^{φ}_{ψ} *denote the formula obtained by replacing (some or all)* φ *inside* η *by* ψ.

If $\varphi \vdash \psi$ *and* $\psi \vdash \varphi$ *then* $\eta \vdash \eta^{\varphi}_{\psi}$ *and* $\eta^{\varphi}_{\psi} \vdash \eta$.

Solution. By induction on the structure of η.

Lemma 3.2.9. *If* $C(\varphi, x, y)$ *and* $\Gamma \vdash \psi$, *then* $\Gamma \vdash \psi^{\forall x\varphi}_{\forall y S^x_y\varphi}$.

Solution. An immediate result of (Ren) and (RS).

Theorem 3.2.10. *(GenC) If* $\Gamma \vdash S^x_a\varphi$ *where* a *does not occur in* $\Gamma \cup \{\varphi\}$, *then* $\Gamma \vdash \forall x\varphi$.

3.3 Semantics of FOL

To give semantics of terms/formulas of first order logic, we need an appropriate structure in which interpret the functions and predicates of FOL.

Definition 3.3.1. *A Tarski structure is a pair $\mathscr{I} = \langle \mathcal{D}, \mathcal{I} \rangle$, where:*

- *$\mathcal{D}$ is a non-empty set, called the domain.*
- *For each n-ary function f, we have $\mathcal{I}(f) \in \mathcal{D}^n \to \mathcal{D}$.*
- *For each n-ary predicate P, we have $\mathcal{I}(P) \in \mathcal{D}^n \to \{0, 1\}$.*

Thus, for each constant a, we have $\mathcal{I}(a) \in \mathcal{D}$.

Definition 3.3.2. *Given a Tarski structure $\mathscr{I} = \langle \mathcal{D}, \mathcal{I} \rangle$, an assignment σ under $\mathscr{I}$ is a mapping $\sigma \colon VS \to \mathcal{D}$.*

We use $\Sigma_{\mathscr{I}}$ to denote the set consisting of assignments under $\mathscr{I}$.

Definition 3.3.3. *Let $\mathscr{I} = \langle \mathcal{D}, \mathcal{I} \rangle$ and $\sigma \in \Sigma_{\mathscr{I}}$.*

Each term t is interpreted to an element $\mathscr{I}(t)(\sigma)$ belonging to $\mathcal{D}$:

- *If $t = x$ is a variable, then $\mathscr{I}(t)(\sigma) = \sigma(x)$.*
- *If $t = f(t_1, \ldots, t_n)$ where f is an n-ary function, then $\mathscr{I}(t)(\sigma) = \mathcal{I}(f)(\mathscr{I}(t_1)(\sigma), \ldots, \mathscr{I}(t_n)(\sigma))$.*

Thus, if $t = a$ is a constant, then $\mathscr{I}(t)(\sigma) = \mathcal{I}(a)$.

Definition 3.3.4. *Each formula φ has a truth value $\mathscr{I}(\varphi)(\sigma) \in \{0, 1\}$:*

- *If $\varphi = P(t_1, \ldots, t_n)$, where P is an n-ary predicate, then $\mathscr{I}(\varphi)(\sigma) = \mathcal{I}(P)(\mathscr{I}(t_1)(\sigma), \ldots, \mathscr{I}(t_n)(\sigma))$.*
- *If $\varphi = \neg\psi$, then $\mathscr{I}(\varphi)(\sigma) = 1 - \mathscr{I}(\psi)(\sigma)$.*

- *If* $\varphi = \psi \to \eta$, *then*

$$\mathscr{I}(\varphi)(\sigma) = \begin{cases} 1 & \text{if } \mathscr{I}(\psi)(\sigma) = 0 \text{ or } \mathscr{I}(\eta)(\sigma) = 1, \\ 0 & \text{if } \mathscr{I}(\psi)(\sigma) = 1 \text{ and } \mathscr{I}(\eta)(\sigma) = 0. \end{cases}$$

- *If* $\varphi = \forall x \psi$, *then*

$$\mathscr{I}(\varphi)(\sigma) = \begin{cases} 1 & \text{if } \mathscr{I}(\psi)(\sigma[x/d]) = 1 \text{ for each } d \in \mathcal{D}, \\ 0 & \text{if } \mathscr{I}(\psi)(\sigma[x/d]) = 0 \text{ for some } d \in \mathcal{D} \end{cases}$$

  where $\sigma[x/d]$ *is a new assignment defined as*

$$\sigma[x/d](y) = \begin{cases} \sigma(y) & \text{if } y \neq x, \\ d & \text{if } y = x. \end{cases}$$

We write $(\mathscr{I}, \sigma) \Vdash \varphi$ *if* $\mathscr{I}(\varphi)(\sigma) = 1$.

Theorem 3.3.5 (Theorem of Substitution)**.** *Suppose that* t *is substitutable for* x *within* φ, *then*

$$(\mathscr{I}, \sigma) \Vdash S_t^x \varphi \text{ if and only if } (\mathscr{I}, \sigma[x/\mathscr{I}(t)(\sigma)]) \Vdash \varphi.$$

We say that $\mathscr{I}$ is a model of φ, denoted as $\mathscr{I} \Vdash \varphi$, if $(\mathscr{I}, \sigma) \Vdash \varphi$ for each $\sigma \in \Sigma_{\mathscr{I}}$.

In particular, we say that $\mathscr{I} = \langle \mathcal{D}, \mathcal{I} \rangle$ is a frugal model of φ if $|\mathcal{D}|$ is not more than the cardinality of the language.

Recall that φ is a sentence, if there is no free variable occurring in φ.

Theorem 3.3.6. *If* φ *is a sentence, then*

- $\mathscr{I} \Vdash \varphi$ *iff* $(\mathscr{I}, \sigma) \Vdash \varphi$ *for some* $\sigma \in \Sigma_{\mathscr{I}}$.

Definition 3.3.7. *Let* φ, ψ *be FOL formulas and* Γ *be a set of FOL formulas. Then we define:*

- $(\mathscr{I}, \sigma) \Vdash \Gamma$ *if for each* $\eta \in \Gamma$, $(\mathscr{I}, \sigma) \Vdash \eta$*;*
- $\Gamma \models \varphi$ *if for each* $\mathscr{I}$ *and* $\sigma \in \Sigma_{\mathscr{I}}$, $(\mathscr{I}, \sigma) \Vdash \Gamma$ *implies* $(\mathscr{I}, \sigma) \Vdash \varphi$*;*
- φ *and* ψ *are equivalent if* $\{\varphi\} \models \psi$ *and* $\{\psi\} \models \varphi$*;*
- φ *is valid if* $\emptyset \models \varphi$*.*

Definition 3.3.8 (Tautology for FOL)**.** *For a formula* $\varphi \in FOF$*, we construct* φ' *as follows:*

- *for each sub-formula* ψ *of* φ *which is either an atomic formula, or a formula of the form* $\forall x \eta$*, we replace it with a corresponding propositional variable* p_ψ*.*

If φ' *is a tautology in propositional logic, then we say* φ *is a tautology for FOL.*

3.4 A Sound and Complete Axiomatization for FOL without Equality $\approx$

3.4.1 The Axiom System: Soundness

Similarly to propositional logic, for FOL we have the soundness property:

Theorem 3.4.1. *If* $\Gamma \vdash \varphi$*, then* $\Gamma \models \varphi$*.*

Hint. For proving the theorem, show and make use of the following results:

- $\{\forall x(\varphi \to \psi), \forall x \varphi\} \models \forall x \psi$;
- if x is not free in φ, then $\vdash \varphi \to \forall x \varphi$. $\square$

Corollary 3.4.2. *If* $\vdash \varphi$*, then* $\models \varphi$*.*

3.4.2 The Axiom System: Completeness

A Hintikka set Γ is a set of FOL formulas fulfilling the following properties:

1. For each atomic formula φ (i.e, $\varphi = P(t_1, \ldots, t_n)$, where $n \geq 0$), either $\varphi \notin \Gamma$ or $\neg\varphi \notin \Gamma$.
2. $\varphi \to \psi \in \Gamma$ implies that either $\neg\varphi \in \Gamma$ or $\psi \in \Gamma$.
3. $\neg\neg\varphi \in \Gamma$ implies that $\varphi \in \Gamma$.
4. $\neg(\varphi \to \psi) \in \Gamma$ implies that $\varphi \in \Gamma$ and $\neg\psi \in \Gamma$.
5. $\forall x\varphi \in \Gamma$ implies that $S_t^x\varphi \in \Gamma$ for each t which is substitutable for x within φ.
6. $\neg\forall x\varphi \in \Gamma$ implies that there is some t with $C(\varphi, x, t)$ such that $\neg S_t^x\varphi \in \Gamma$.

Note: $C(\varphi, x, t)$ iff $C(\varphi, x, y)$ for all y occurring in t.

Lemma 3.4.3. *A Hintikka set Γ is consistent, and moreover, for each formula φ, either $\varphi \notin \Gamma$, or $\neg\varphi \notin \Gamma$.*

Theorem 3.4.4. *A Hintikka set Γ is satisfiable, i.e, there is some interpretation $\mathscr{I}$ and some $\sigma \in \Sigma_{\mathscr{I}}$ such that $(\mathscr{I}, \sigma) \Vdash \varphi$ for each $\varphi \in \Gamma$.*

Theorem 3.4.5. *If Γ is a set of FOL formulas, then "Γ is consistent" implies that "Γ is satisfiable".*

Particularly, if Γ consists only of sentences, then Γ has a frugal model.

Proof. Let us enumerate[1] the formulas as $\varphi_0, \varphi_1, \ldots, \varphi_n, \ldots$, and subsequently define a series of formula sets as follows. Let $\Gamma_0 = \Gamma$, and

$$\Gamma_{i+1} = \begin{cases} \Gamma_i \cup \{\neg\varphi_i\} & \text{if } \Gamma_i \vdash \neg\varphi_i \\ \Gamma_i \cup \{\varphi_i\} & \text{if } \Gamma_i \nvdash \neg\varphi_i \text{ and } \varphi_i \neq \neg\forall x\psi \\ \Gamma_i \cup \{\varphi_i, \neg S_a^x\psi\} & \text{if } \Gamma_i \nvdash \neg\varphi_i, \text{ and } \varphi_i = \neg\forall x\psi \end{cases}$$

Above, for each formula $\forall x\psi$, we pick and fix the constant a which does not occur in $\Gamma_i \cup \{\varphi_i\}$.

Finally let $\Gamma^* = \lim_{i\to\infty} \Gamma_i$.

If Γ is consistent, the set Γ^* is maximal and consistent, and is referred to as the Henkin set. Thus, a Henkin set is also a Hintikka set. □

Theorem 3.4.6. *If $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$.*

Corollary 3.4.7. *If $\models \varphi$, then $\vdash \varphi$.*

Theorem 3.4.8. *Γ is consistent iff each of its finite subset is consistent. Moreover, Γ is satisfiable iff each of its finite subsets is satisfiable.*

3.5 A Sound and Complete Axiomatization for FOL with Equality $\approx$

The axiomatization based on the Hilbert's systems seen in the previous section can be extended to the case of first order logic with the equality $\approx$. To do this, two additional axioms have to be included in the Hilbert's system:

[1] We assume the language to be countable, yet the result can be extended to languages with arbitrary cardinality.

$A_{\approx}$**:** $x \approx x$;

$A'_{\approx}$**:** $(x \approx y) \to (\alpha \to \alpha^x_y)$, where α is an atomic formula.

The soundness and completeness results can be proved similarly in the extended Hilbert's system; note that for the completeness one, a variation of the Tarski structure is required, namely, the domain considered in the construction modulo the relation $\approx$. This allows us so manage correctly the formulas that are equivalent under $\approx$.

The actual details about the above construction are omitted; the interested reader is invited to formalize them.