- *If* $\varphi = \neg\psi$, *then* $\mathscr{I}(\varphi)(\sigma) = 1 - \mathscr{I}(\psi)(\sigma)$.
- *If* $\varphi = \psi \to \eta$, *then*

$$\mathscr{I}(\varphi)(\sigma) = \begin{cases} 1 & \text{if } \mathscr{I}(\psi)(\sigma) = 0 \text{ or } \mathscr{I}(\eta)(\sigma) = 1, \\ 0 & \text{if } \mathscr{I}(\psi)(\sigma) = 1 \text{ and } \mathscr{I}(\eta)(\sigma) = 0. \end{cases}$$

- *If* $\varphi = \forall x\psi$, *then*

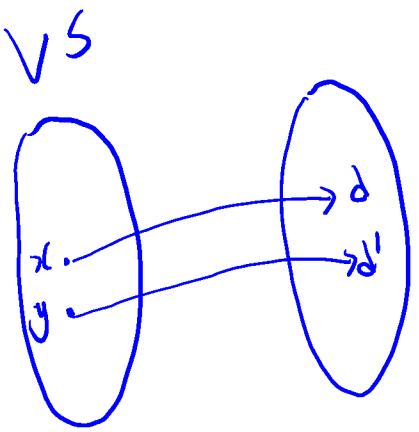
$$\mathscr{I}(\varphi)(\sigma) = \begin{cases} 1 & \text{if } \mathscr{I}(\psi)(\sigma[x/d]) = 1 \text{ for each } d \in \mathcal{D}, \\ 0 & \text{if } \mathscr{I}(\psi)(\sigma[x/d]) = 0 \text{ for some } d \in \mathcal{D} \end{cases}$$

where $\sigma[x/d]$ *is a new assignment defined as*

$$\sigma[x/d](y) = \begin{cases} \sigma(y) & \text{if } y \neq x, \\ d & \text{if } y = x. \end{cases}$$

$\sigma[x/d] : VS \to D$

We write $(\mathscr{I}, \sigma) \Vdash \varphi$ *if* $\mathscr{I}(\varphi)(\sigma) = 1$.



It holds:

$(\sigma[x/d])[y/d'] = (\sigma[y/d'])[x/d]$

$$\Rightarrow \begin{cases} I(f)(T(s^{\mathcal{X}}_f t_1')(\sigma) \ldots T(s^{\mathcal{X}}_f t_k')(\sigma)) \\ I(f)(T(t_1')(\sigma[x/T(t)(\sigma)]) \ldots T(t_k')(\sigma[x/T(t)(\sigma)])) \end{cases}$$

IH || IH ||

Lemma: $T(\underline{S^x_t t_i})(\sigma) = T(t_i)(\sigma[x/T(t)(\sigma)])$ $\forall t_i \in T$

— $t_i = y$ ① $x = y$ $T(t)(\sigma) = \underline{T(y)(\sigma[x/T(t)(\sigma)])}$

$S^x_t t_i = t$

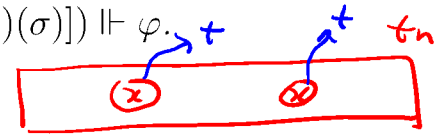
② $x \neq y$ $T(y)(\sigma) = T(y)(\sigma[x/\cdots)) = \sigma(y)$

— $t_i = f(t_1' \cdots t_k')$ $\left.\begin{array}{l} T(f(S^x_t t_1' \cdots S^x_t t_k'))(\sigma) \\ = T(f(t_1' \cdots t_k'))(\sigma[x/T(t)(\sigma)]) \end{array}\right] \Rightarrow$

Theorem 1.3.5 (Theorem of Substitution). *Suppose that t is substitutable for x within φ, then*

条件

$$(\mathscr{I}, \sigma) \Vdash S^x_t\varphi \textit{ if and only if } (\mathscr{I}, \sigma[x/\mathscr{I}(t)(\sigma)]) \Vdash \varphi.$$



$\varphi = P(t_1 \cdots t_n)$ $\quad = P(\underline{S^x_t t_1} \cdots \underline{S^x_t t_n})$

$(T, \sigma) \Vdash \underline{S^x_t P(t_1 \cdots t_n)}$ iff $(T, \sigma[x/T(t)(\sigma)]) \Vdash P(t_1 \cdots t_n)$

lhs $:= I(P)(T(S^x_t t_1)(\sigma) \cdots T(S^x_t t_n)(\sigma)) = 1$

$\quad\quad\quad\quad\quad\quad \parallel \quad\quad\quad\quad\quad\quad\quad \parallel$

rhs $:= I(P)(T(t_1)(\sigma[x/T(t)(\sigma)]), \cdots T(t_n)(\sigma[x/T(t)(\sigma)])) = 1$

$\varphi = \neg\psi$ $\quad = \neg S^x_t \psi$

$(T, \sigma) \Vdash S^x_t \neg\psi$ iff $(T, \sigma) \nVdash S^x \psi$ $\overset{IH}{\text{iff}}$ $(T, \sigma[x/\cdots]) \nVdash \psi$

iff $(T, \sigma[x/\cdots]) \Vdash \varphi$

$\varphi = \psi \to \eta$ ✓

$\varphi = \forall y \psi$ $(T, \sigma) \Vdash S^x_t \forall y \psi$ iff $(T, \sigma[x/T(t)(\sigma)]) \Vdash \forall y \psi$

① $y = x$ $(T, \sigma) \Vdash \forall y \psi$ iff $(T, \sigma[x/T(t)(\sigma)]) \Vdash \forall y \psi$

$\forall d$: $(T, \sigma[y/d]) \Vdash \psi$ iff $\forall d$: $(T, [\sigma[x/\underline{T(t)(\sigma)}]][y/d]) \Vdash \psi$

② $y \neq x$

lhs: $(T, \sigma) \Vdash \forall y (S^x_t \psi)$ iff $\forall d$ $(T, \sigma[y/d]) \Vdash S^x_t \psi$

$\parallel$ IH

rhs: $\forall d$ $(T, (\sigma[x/T(t)(\sigma)])[y/d]) \Vdash \psi$ iff $\forall d$ $(T, \sigma[y/d])[x/T(t)(\sigma)]) \Vdash \psi$

$\Gamma \cup \{\neg\forall x\psi \to \neg S^x_a \psi\}$ is consistent.

Assume it is not, then $\Gamma \vdash \neg(\neg\forall x\psi \to \neg S^x_a\psi) \equiv \neg\forall x\psi \wedge S^x_a\psi$

$\Rightarrow \Gamma \vdash \neg\forall x\psi$ and $\underbrace{\Gamma \vdash S^x_a\psi}$ $\overset{\text{Gen}}{\Longrightarrow} \Gamma \vdash \forall x\psi$ ↯

- We first add all Henkin formulas into Γ.

Theorem 1.4.5. *If Γ is a set of FOL formulas, then "Γ is consistent" implies that "Γ is satisfiable".*

Particularly, if Γ consists only of sentences, then Γ has a frugal model.

Proof. Let us enumerate[1] the formulas as $\varphi_0, \varphi_1, \ldots, \varphi_n, \ldots$, and subsequently define a series of formula sets as follows. Let $\Gamma_0 = \Gamma$, and

Verify is consistent.

$$\Gamma_{i+1} = \begin{cases} \Gamma_i \cup \{\neg\varphi_i\} & \text{if } \Gamma_i \vdash \neg\varphi_i \\ \Gamma_i \cup \{\varphi_i\} & \text{if } \Gamma_i \nvdash \neg\varphi_i \text{ and } \varphi_i \neq \neg\forall x\psi \\ \underbrace{\Gamma_i \cup \{\varphi_i, \neg S^x_a\psi\}} & \text{if } \Gamma_i \nvdash \neg\varphi_i, \text{ and } \varphi_i = \neg\forall x\psi \end{cases}$$

Above, for each formula $\forall x\psi$, we pick and fix the constant $\underline{a}$ which does not occur in $\Gamma_i \cup \{\varphi_i\}$.

- $\Gamma^* \vdash \varphi$ iff $\varphi \in \Gamma^*$
- $\Gamma^* \models \varphi$ iff $\varphi \in \Gamma^*$

Henkin formula

Henkin Lemma: $\vdash \boxed{\neg\forall x\psi \to \neg S^x_a\psi}$ where $a \notin Oc(\psi)$

A3 $\vdash S^x_a\psi \to \forall x\psi$

L_V $\vdash S^x_a\psi \overset{\text{Conc}}{\Longrightarrow} \vdash \forall x\psi$

L_V: Assume $\Gamma \vdash \varphi$ implies $\Gamma \vdash \psi$. Then $\Gamma \vdash \varphi \to \psi$

[1] We assume the language to be countable, yet the result can be extended to languages with arbitrary cardinality.

Proof: $\Gamma, \varphi \vdash \varphi \overset{\text{assumption}}{\Longrightarrow} \Gamma, \varphi \vdash \psi$

$\Longrightarrow \Gamma \vdash \varphi \to \psi$

D: set of all terms

$I(f)(t_1 \cdots t_n) = f(t_1 \cdots t_n)$

$\boxed{I(P)(t_1 \cdots t_n) = 1 \text{ iff } P(t_1 \cdots t_n) \in \Gamma^*}$

$\sigma(x) = x. \quad \Rightarrow \sigma(t) = t$

Finally let $\Gamma^* = \lim_{i \to \infty} \Gamma_i$.

If Γ is consistent, the set Γ^* is maximal and consistent, and is referred to as the Henkin set. Thus, a Henkin set is also a Hintikka set. $\square$

$(\mathcal{T}, \sigma) \models \varphi$ iff $\varphi \in \Gamma^*$

- $\varphi = P(t_1 \cdots t_n)$

- $\varphi = \neg\psi \quad (\mathcal{T}, \sigma) \models \neg\psi$ iff $(\mathcal{T}, \sigma) \not\models \psi$ $\overset{IH}{\text{iff}}$ $\psi \notin \Gamma^*$ iff $\varphi \in \Gamma^*$

- $\varphi = \psi \to \eta$ ✓

- $\varphi = \forall x \psi \quad (\mathcal{T}, \sigma) \models \forall x \psi$ iff $\forall x \psi \in \Gamma^*$

$\Rightarrow (\mathcal{T}, \sigma) \models \forall x \psi \Rightarrow (\mathcal{T}, \sigma) \models S^x_a \psi \overset{IH}{\Longrightarrow} S^x_a \psi \in \Gamma^* \Rightarrow \neg S^x_a \psi \notin \Gamma^*$

$\Rightarrow \neg \forall x \psi \notin \Gamma^* \Rightarrow \forall x \psi \in \Gamma^*$

$\Leftarrow (\mathcal{T}, \sigma) \not\models \forall x \psi \Rightarrow \exists t : (\mathcal{T}, \sigma[x/t]) \not\models \psi$

$\boxed{\text{Pick } \eta : \ \eta \vdash \neg\psi, \ t \text{ is substitutable for } x \text{ in } \eta}$

$\Rightarrow (\mathcal{T}, \sigma[x/t]) \not\models \eta$

$\Rightarrow (\mathcal{T}, \sigma) \not\models S^x_t \eta \overset{IH}{\Rightarrow} S^x_t \eta \notin \Gamma^*$

$\Rightarrow (\mathcal{T}, \sigma) \not\models S^x_t \psi \Rightarrow \forall x \eta \notin \Gamma^*$

$\overset{IH}{\Rightarrow} S^x_t \psi \notin \Gamma^*$

$\Rightarrow \forall x \psi \notin \Gamma^*$

$\Rightarrow \forall x \psi \notin \Gamma^*$