Steps ③ can be omitted by constructing the sequence:

$$\forall y\, S^a_y \varphi_0 \quad \forall y\, S^a_y \varphi_1 \quad \cdots \quad \forall y\, S^a_y \varphi_n$$

Exercise: prove it is a deductive sequence

Theorem 3.2.10. *(GenC) If $\Gamma \vdash S^x_a \varphi$ where a does not occur in $\Gamma \cup \{\varphi\}$, then $\Gamma \vdash \forall x \varphi$.*

① $\Gamma \vdash S^x_a \varphi \Rightarrow \exists\ \underbrace{\varphi_0\ \varphi_1\ \cdots\ \varphi_n}_{\varphi_n = S^x_a \varphi}$ proof sequence

Construct Γ_0: set of $\varphi_i \in \Gamma$ $i \in \{0, \cdots n\}$ $\Rightarrow \Gamma_0 \vdash S^x_a \varphi$

Obser: $\underline{\Gamma_0 \vdash \forall x \varphi} \Longrightarrow \Gamma \vdash \forall x \varphi$

② Pick y such that $y \notin Occ(\Gamma_0, \forall x \varphi)$

Obser: $\underline{\Gamma_0 \vdash \forall y S^x_y \varphi} \Rightarrow \Gamma_0 \vdash \forall x \varphi$ $\quad \forall y S^x_y \varphi \vdash\dashv \forall x \varphi$

③ $\underline{\Gamma_0 \vdash S^x_y \varphi} \Rightarrow \Gamma_0 \vdash \forall y S^x_y \varphi$ [Gen]

④ Construct: $S^a_y \varphi_0\ S^a_y \varphi_1\ \cdots\ S^a_y \varphi_n$ $\quad S^a_y(S^x_a \varphi) = \boxed{S^a_y \varphi}$

Induction over $\underline{\underline{n}}$

$n = 0$:

$\varphi_0 \in A_{1\cdots 7}$: $\Rightarrow S^a_y \varphi_0 \in A_{1\cdots 7}$

$\varphi_0 \in \Gamma_0 \Rightarrow \underbrace{S^a_y \varphi_0}_{= \varphi_0} \in \Gamma_0$ (as $a \notin Occ(\Gamma_0)$)

$n-1 \to n$

$\varphi_i \xrightarrow{IH} S^a_y \varphi_i$

$\varphi_j = \varphi_i \to \varphi_n \qquad S^a_y(\varphi_i \to \varphi_n)$

$\qquad\qquad\qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad\qquad\qquad\qquad S^a_y \varphi_i \to S^a_y \varphi_n$

$mp \Rightarrow S^a_y \varphi_n$

φ $\boxed{\underset{t}{x} \quad \underset{t}{x} \quad \underset{t}{x}}$

$\sigma[x/\mathcal{T}(t)(\sigma)](y) = \begin{cases} \sigma(y) & \text{if } x \neq y \\ \mathcal{T}(t)(\sigma) & \text{otherwise} \end{cases}$

Theorem 3.3.5 (Theorem of Substitution)**.** *Suppose that t is substitutable for x within φ, then* Prove by structural induction over φ

$$(\mathscr{I}, \sigma) \Vdash S_t^x \varphi \text{ if and only if } (\mathscr{I}, \sigma[x/\mathscr{I}(t)(\sigma)]) \Vdash \varphi.$$ $\Rightarrow$ Exercise!

We say that $\mathscr{I}$ is a model of φ, denoted as $\mathscr{I} \Vdash \varphi$, if $(\mathscr{I}, \sigma) \Vdash \varphi$ for each $\sigma \in \Sigma_{\mathscr{I}}$.

In particular, we say that $\mathscr{I} = \langle \mathcal{D}, \mathcal{I} \rangle$ is a frugal model of φ if $|\mathcal{D}|$ is not more than the cardinality of the language. Stated differently, it holds $|D| \leq |FOF|$

Recall that φ is a sentence, if there is no free variable occurring in φ.

Theorem 3.3.6. *If φ is a sentence, then*

- $\mathscr{I} \Vdash \varphi$ *iff* $(\mathscr{I}, \sigma) \Vdash \varphi$ *for some* $\sigma \in \Sigma_{\mathscr{I}}$.

Definition 3.3.7. *Let φ, ψ be FOL formulas and Γ be a set of FOL formulas. Then we define:*

- $(\mathscr{I}, \sigma) \Vdash \Gamma$ *if for each* $\eta \in \Gamma$, $(\mathscr{I}, \sigma) \Vdash \eta$;
- $\Gamma \models \varphi$ *if for each* $\mathscr{I}$ *and* $\sigma \in \Sigma_{\mathscr{I}}$, $(\mathscr{I}, \sigma) \Vdash \Gamma$ *implies* $(\mathscr{I}, \sigma) \Vdash \varphi$;
- φ *and* ψ *are equivalent if* $\{\varphi\} \models \psi$ *and* $\{\psi\} \models \varphi$;

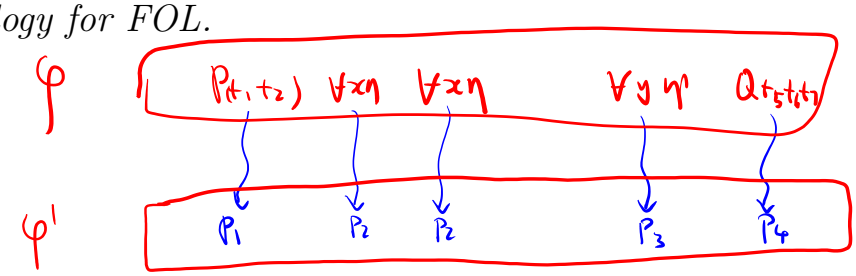
普遍有效

- φ *is valid if* $\emptyset \models \varphi$.

Definition 3.3.8 (Tautology for FOL)**.** *For a formula* $\varphi \in$ *FOF, we construct* φ' *as follows:*

- *for each sub-formula* ψ *of* φ *which is either an atomic formula, or a formula of the form* $\forall x\eta$*, we replace it with a corresponding propositional variable* p_ψ*.*

If φ' *is a tautology in propositional logic, then we say* φ *is a tautology for FOL.*



By construction: φ' is a propositional formula.

- $\forall x\, Px \to \overset{S^x_x Px}{Px}$ is valid

  $p_1 \to p_2$ is not a tautology.

- $\forall x\, P(x,y) \to \forall x\, P(x,y)$ is valid, but not a tautology.

- $(\underbrace{P(x)}_{p_1} \to \underbrace{\forall x\, Q(x,y)}_{p_2}) \to (\neg \underbrace{\forall x\, Q(x,y)}_{p_2} \to \neg \underbrace{P(x)}_{p_1})$ is a tautology [A3]

- φ is tautology $\Rightarrow$ φ is valid, but not the other direction.

3.4 A Sound and Complete Axiomatization for FOL without Equality $\approx$

3.4.1 The Axiom System: Soundness

Similarly to propositional logic, for FOL we have the soundness property:

Theorem 3.4.1. *If* $\Gamma \vdash \varphi$*, then* $\Gamma \models \varphi$*.*

Hint. For proving the theorem, show and make use of the following results:

- $\{\forall x(\varphi \to \psi), \forall x\varphi\} \models \forall x\psi$;
- if x is not free in φ, then $\vdash \varphi \to \forall x\varphi$. $\square$

Corollary 3.4.2. *If* $\vdash \varphi$*, then* $\models \varphi$*.*

$\Gamma \vdash \varphi \Rightarrow \exists\ \varphi_0\ \varphi_1\ \cdots\ \varphi_n = \varphi$

$n=0$: $\varphi_0 \in \Gamma$: $(T,\sigma) \models \Gamma \Rightarrow (T,\sigma) \models \varphi_0$

$\varphi_0 \in Axs$: ✓

φ_0: $\forall x\varphi \to S^x_t \varphi$

- $(T,\sigma) \not\models \forall x\varphi$ ✓
- Assume $(T,\sigma) \models \forall x\varphi$.

By Def: $\forall d \in D$: $(T, \sigma[x/d]) \models \varphi$

pick: $d = T(\underset{\sim}{t})(\sigma)$

By Theorem of Sub: $(T, \sigma) \models S^x_t \varphi$

$\varphi_0 = \forall x(\varphi \to \psi) \to (\forall x\varphi \to \forall x\psi)$

- Assume $(T,\sigma) \models \{\forall x(\varphi \to \psi), \forall x\varphi\}$

$\Rightarrow \forall d$: $(T, \sigma[x/d]) \models \varphi \to \psi$, φ $\Big\} \Rightarrow \psi$

$\varphi_0 = \varphi \to \forall x\varphi$ $\quad x \notin free(\varphi)$

- Assume $(T,\sigma) \models \varphi$.

$\Downarrow$

$\forall d \in D$: $(T, \sigma[x/d]) \models \varphi \Leftrightarrow \cdots$

$\boxed{n-1 \Rightarrow n}$

- $\varphi_n \in \Gamma$
- $\varphi_n \in Ax\cdots$

} same as base case

- MP: same as propositional logic.

3.4.2 The Axiom System: Completeness

A Hintikka set Γ is a set of FOL formulas fulfilling the following properties:

1. For each atomic formula φ (i.e, $\varphi = P(t_1, \ldots, t_n)$, where $n \geq 0$), either $\varphi \notin \Gamma$ or $\neg\varphi \notin \Gamma$.
2. $\overset{\neg\varphi \vee \psi}{\varphi \to \psi} \in \Gamma$ implies that either $\neg\varphi \in \Gamma$ or $\psi \in \Gamma$.
3. $\neg\neg\varphi \in \Gamma$ implies that $\varphi \in \Gamma$.
4. $\overset{\varphi \wedge \neg\psi}{\neg(\varphi \to \psi)} \in \Gamma$ implies that $\varphi \in \Gamma$ and $\neg\psi \in \Gamma$.
5. $\forall x\varphi \in \Gamma$ implies that $S^x_t\varphi \in \Gamma$ for each t which is substitutable for x within φ.
6. $\overset{\exists x \neg\varphi}{\neg\forall x\varphi} \in \Gamma$ implies that there is some t with $C(\varphi, x, t)$ such that $\neg S^x_t\varphi \in \Gamma$.

Note: $C(\varphi, x, t)$ iff $C(\varphi, x, y)$ for all y occurring in t.

Lemma 3.4.3. *A Hintikka set Γ is consistent, and moreover, for each formula φ, either $\varphi \notin \Gamma$, or $\neg\varphi \notin \Gamma$.*

By structural induction over φ.

$\Rightarrow$ Exercise

Theorem 3.4.4. *A Hintikka set Γ is satisfiable, i.e, there is some interpretation $\mathscr{I}$ and some $\sigma \in \Sigma_{\mathscr{I}}$ such that $(\mathscr{I}, \sigma) \Vdash \varphi$ for each $\varphi \in \Gamma$.*

Theorem 3.4.5. *If Γ is a set of FOL formulas, then "Γ is consistent" implies that "Γ is satisfiable".*

Particularly, if Γ consists only of sentences, then Γ has a frugal model.

Proof. Let us enumerate[1] the formulas as $\varphi_0, \varphi_1, \ldots, \varphi_n, \ldots$, and subsequently define a series of formula sets as follows. Let $\Gamma_0 = \Gamma$, and

$\Gamma \cup \varphi$ consistent iff $\Gamma \nvdash \neg\varphi$

$$\Gamma_{i+1} = \begin{cases} \Gamma_i \cup \{\neg\varphi_i\} & \text{if } \Gamma_i \vdash \neg\varphi_i \\ \Gamma_i \cup \{\varphi_i\} & \text{if } \Gamma_i \nvdash \neg\varphi_i \text{ and } \varphi_i \neq \neg\forall x\psi \\ \Gamma_i \cup \{\varphi_i, \neg S^x_a \psi\} & \text{if } \Gamma_i \nvdash \neg\varphi_i, \text{ and } \varphi_i = \neg\forall x\psi \end{cases}$$

$\exists x \neg\psi$

Above, for each formula $\forall x\psi$, we pick and fix the constant a which does not occur in $\Gamma_i \cup \{\varphi_i\}$.

Finally let $\Gamma^* = \lim_{i\to\infty} \Gamma_i$.

If Γ is consistent, the set Γ^* is maximal and consistent, and is referred to as the Henkin set. Thus, a Henkin set is also a Hintikka set. $\square$

We read the model $T = (D, I)$ from Γ^* as follows:

- D is the set of terms Terms
- $I(f) : D^n \longrightarrow D$
  $(t_1, t_2, \cdots t_n) \longmapsto f(t_1, \cdots t_n)$
- $I(P) : D^n \longrightarrow \{0,1\}$

[1] We assume the language to be countable, yet the result can be extended to languages with arbitrary cardinality.

$I(P)(t_1, t_2, \cdots t_n) = 1$ iff $P(t_1, \cdots t_n) \in \Gamma^*$

$\sigma : VS \longrightarrow D$ $\sigma(x) = x$

Remains: $(T, \sigma) \Vdash \Gamma^* \iff \forall \varphi \in \Gamma^* : (T, \sigma) \Vdash \varphi$

structural induction over φ

Theorem 3.4.6. *If* $\Gamma \models \varphi$*, then* $\Gamma \vdash \varphi$*.*

Corollary 3.4.7. *If* $\models \varphi$*, then* $\vdash \varphi$*.*

Theorem 3.4.8. Γ *is consistent iff each of its finite subset is consistent. Moreover,* Γ *is satisfiable iff each of its finite subsets is satisfiable.*

Same as propositional logic

3.5 A Sound and Complete Axiomatization for FOL Equality $\approx$

with

We need two additional axioms:

$A_{\approx}$ $\quad x \approx x$

$A_{\approx}'$ $\quad x \approx y \to (\alpha \to \alpha^{x}_{y})$

α is atomic formula.

Overview: