Discrete Mathematics[1]
http://iscasmc.ios.ac.cn/DM2016

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March 17, 2016

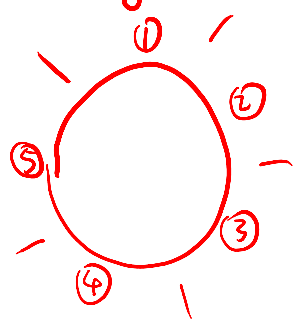
[1]Materials about the axiomatization are provided by Dr. Wanwei Liu.

Contents

- theory
- mathema.
- reuse
- mathematical lang.

Friedrich Frege

«Begriffsschrift»



Chapter 3

First Order Logic (FOL)

3.1 Syntax of FOL

Propositional logic is a coarse language, which only concerns about propositions and boolean connectives. Practically, this logic is not powerful enough to describe important properties we are interested in.

Example 3.1.1 (Syllogism of Aristotle)**.** *Consider the following assertions:*

1. *All men are mortal.*
2. *Socrates is a man.*
3. *So Socrates would die.*

个体 谓词

$$\forall x(Man(x) \to Mortal(x))$$

Definition 3.1.2. *First order logic is an extension of proposition logic:*

1. *To accept parameters, it generalized propositions to predicates.*

2. *To designate elements in the domain, it is equipped with functions and constants.*

3. *It also involves quantifiers to capture infinite conjunction and disjunction.*

$x > 3$ $\quad$ $> x 3$

Definition 3.1.3. *We are given:*

变元

- *an arbitrary set of variable symbols* $VS = \{x, y, x_1, \dots\}$;
- *an arbitrary set (maybe empty) of function symbols* $FS = \{f, g, f_1, \dots\}$, *where each symbol has an arity;* 函词
- *an arbitrary set (maybe empty) of predicate symbols* $PS = \{P, Q, P_1, \dots\}$, *where each symbol has an arity;* 谓词
- *an equality symbol set* ES *which is either empty or one element set containing* $\{\approx\}$. 等词

Let $L = VS \cup \{(,), \to, \neg, \forall\} \cup \boxed{FS \cup PS \cup ES}$. *Here* $VS \cup \{(,), \to, \neg, \forall\}$ *are referred to as* logical symbols, *and* $FS \cup PS \cup ES$ *are referred to as* non-logical symbols.

We often make use of the

0-元函词

- *set of constant symbols, denoted by* $CS = \{a, b, a_1, \dots\} \subseteq FS$, *which consist of function symbols with arity* 0;

→ 0元谓词

- *set of propositional symbols, denoted by* $PS = \{p, q, p_1, \dots\} \subseteq FS$*, which consist of predicate symbols with arity* 0*.*

Definition 3.1.4 (FOL terms)**.** *The terms of the first order logic are constructed according to the following grammar:*

$$t ::= x \mid f t_1 \dots t_n$$

where $x \in VS$*, and* $f \in FS$ *has arity* n*.*

Accordingly, the set T *of terms is the smallest set satisfying the following conditions:* 项

- *each variable* $x \in VS$ *is a term.*
- *Compound terms:* $f t_1 \dots t_n$ *is a term (thus in* T*), provided that* f *is a* n*-arity function symbol, and* $t_1, \dots, t_n \in T$*. Particularly,* $a \in CS$ *is a term.*

We often write $f(t_1, \dots, t_n)$ *for the compound terms.*

$L = \{ \overset{\text{const.}}{0}, \underset{\text{1元函数}}{S}, +, \cdot, <, \approx \}$

$\boxed{\dfrac{S0 < x}{< \boxed{S0}\ \boxed{x}}}$ t_1 t_2 2元

- $Sx \not\approx 0$ $\quad \neg(Sx \approx 0)$
- $x \approx y \leftrightarrow Sx \approx Sy$
- $(\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(Sx))) \to \forall x\, \varphi(x)$
- x is prime

$$S0 < x \wedge \forall y \forall z (y < x \wedge z < x \to y \cdot z \not\approx x)$$

Definition 3.1.5 (FOL formulas)**.** *The well-formed formulas of the first order logic are constructed according to the following grammar:*

$$\varphi ::= Pt_1 \dots t_n \mid \neg\varphi \mid \varphi \to \varphi \mid \forall x \varphi$$

where $t_1, \dots, t_n$ are terms, $P \in PS$ has arity n, and $x \in VS$.

We often write $P(t_1, \dots, t_n)$ for clarity. Accordingly, the set $\boxed{FOF}$ of first order formulas is the smallest set satisfying:

- *$P(t_1, \dots, t_n) \in FOF$ is a formula, referred to as the atomic formula.*
- *Compound formulas: $(\neg\varphi)$ (negation), $(\varphi \to \psi)$ (implication), and $(\forall x \varphi)$ (universal quantification) are formulas (thus in FOF), provided that $\varphi, \psi \in FOF$.*

We omit parentheses if it is clear from the context.

As syntactic sugar, we can define $\exists x \varphi$ as $\exists x \varphi := \neg \forall x \neg \varphi$. We assume that $\forall$ and $\exists$ have higher precedence than all logical operators.

$L_{er} = \{R\}$

- Rxx
- $Rxy \to Ryx$
- $Rxy \wedge Ryz \to Rxz$

$\forall x \varphi \quad \neg \exists x \neg \varphi$

$L_{group} = \{e, +, \approx\}$

- $x + (y + z) \approx (x + y) + z$
- $e + x \approx x$
- $\forall x \exists y (x + y \approx e)$

FOF

Definition 3.1.6 (Sub-formulas)**.** *For a formula φ, we define the sub-formula function $Sf : FOF \to 2^{FOF}$ as follows:*

$$
\begin{aligned}
Sf(P(t_1, \ldots, t_n)) &= \{P(t_1, \ldots, t_n)\} \\
Sf(\neg\varphi) &= \{\neg\varphi\} \cup Sf(\varphi) \\
Sf(\varphi \to \psi) &= \{\varphi \to \psi\} \cup Sf(\varphi) \cup Sf(\psi) \\
Sf(\forall x\varphi) &= \{\forall x\varphi\} \cup Sf(\varphi) \\
Sf(\exists x\varphi) &=
\end{aligned}
$$

scope of $\exists x$

自由

φ $\quad \exists x(x > 5) \quad x < y \quad \forall x(x < y)$

辖域

Definition 3.1.7 (Scope)**.** *The part of a logical expression to which a quantifier is applied is called the scope of this quantifier. Formally, each sub-formula of the form $Qx\psi \in Sf(\varphi)$, the scope of the corresponding quantifier Qx is ψ. Here $Q \in \{\forall, \exists\}$.*

$Occ(x) = \{x\}$

$Occ(f t_1 \cdots t_n) = \bigcup Occ(t_i)$

$Occ(P t_1 \cdots t_n) = \bigcup Occ(t_i)$

$Occ(\neg \varphi) = Occ(\varphi)$

$\varphi \to \psi =$

$\forall x \varphi = \{x\} \cup Occ(\varphi)$

Substitution for Terms

句子

Definition 3.1.8 (Sentence). *We say an occurrence of x in φ is free if it is not in scope of any quantifiers $\forall x$ (or $\exists x$). Otherwise, we say that this occurrence is a bound occurrence. If a variable φ has no free variables, it is called a closed formula, or a sentence.*

自由

约束

闭公式

$free(\varphi)$

$free : FOF \to 2^{VS}$

$free(P t_1 \cdots t_n) = Occ(P t_1 \cdots t_n)$

$free(\neg \varphi) = free(\varphi)$

$free(\varphi \to \psi) = -$

$free(\forall x \varphi) = free(\varphi) \setminus \{x\}$

Definition 3.1.9 (Substitution). *The substitution of x with t within φ, denoted as $S_t^x \varphi$, is obtained from φ by replacing each free occurrence of x with t.*

$S_t^x : T \to T$

$S_t^x(y) = \begin{cases} t & y = x \\ y & \text{other.} \end{cases}$

$S_t^x : FOF \to FOF$

$S_t^x(\varphi)$

$S_t^x P(t_1 \cdots t_n) = P(S_t^x t_1 \cdots S_t^x t_n)$

$S_t^x(\neg \varphi) = \neg S_t^x \varphi$

$S_t^x(\varphi \to \psi) = S_t^x \varphi \to S_t^x \psi$

$S_t^x(\forall y \varphi) = \begin{cases} \forall y \varphi & \text{if } x = y \\ \forall y\, S_t^x \varphi & \text{other.} \end{cases}$

We would extend this notation to $S^{x_1,\dots,x_n}_{t_1,\dots,t_n}\varphi$.

Remark 3.1.10. *It is important to remark that $S^{x_1,\dots,x_n}_{t_1,\dots,t_n}\varphi$ is not the same as $S^{x_1}_{t_1}\dots S^{x_n}_{t_n}\varphi$: the former performs a simultaneous substitution.*

For example, consider the formula $P(x,y)$: the subsitution $S^{x,y}_{y,x}P(x,y)$ gives $S^{x,y}_{y,x}P(x,y) = P(y,x)$ while the substitutions $S^x_y S^y_x P(x,y)$ give $S^x_y S^y_x P(x,y) = S^x_y P(x,x) = P(y,y)$.

$$\left(S^{x_1}_{t_1} \circ S^{x_2}_{t_2} \circ \cdots S^{x_n}_{t_n}\right)(\varphi)$$

$$S^x_t \varphi = \exists y(t<y) \qquad S^x_y \varphi = \exists y(y<y)$$

Remark 3.1.11. *Consider $\varphi = \exists y(x<y)$ in the number theory. What is $S^x_t\varphi$ for the special case of $t = y$?*

Definition 3.1.12 (Substitutable on Terms)**.** *We say that t is substitutable for x within φ iff for each variable y occurring in t, there is no free occurrence of x in scope of $\forall y/\exists y$ in φ.*

$\forall y \in Occ(t)$

φ $\quad \forall y\ \varphi \quad x \quad \forall y \quad \forall y\ \varphi'$

no x!

$\forall x\ \psi$ ✓

$\to \varphi = P(t_1 \cdots t_n)$ $\quad$ t 可替换 x

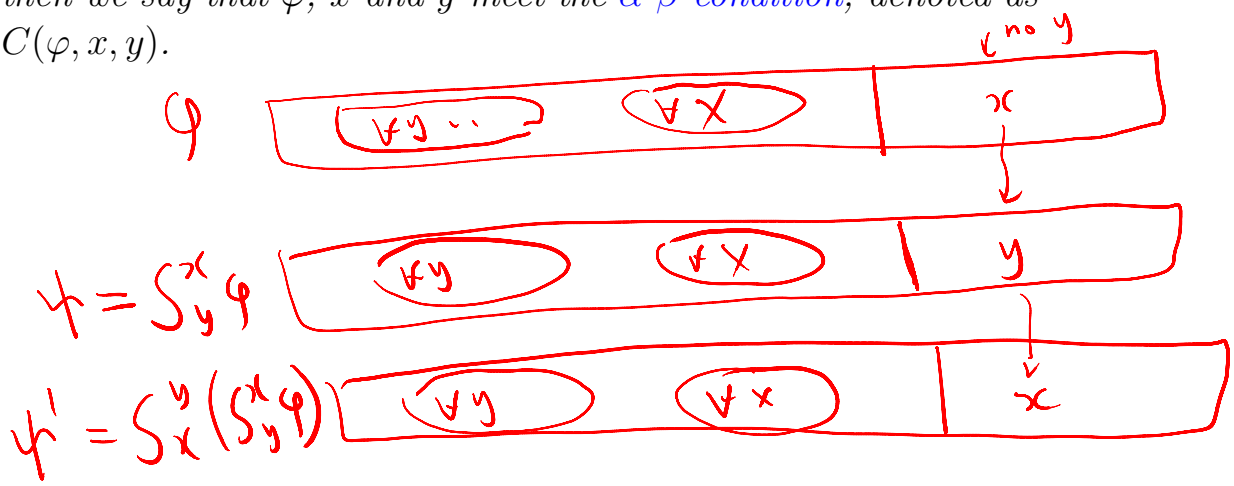
$\to \varphi = \neg\psi$, $\psi \to \eta$

$\to \varphi = \forall y \psi$ $\quad \begin{cases} x \notin free(\varphi) \checkmark \text{ or} \\ y \notin free(t) \wedge \\ \quad t \text{ is sub for } x \text{ in } \psi \end{cases}$

Definition 3.1.13 (α-β condition)**.** *If the formula φ and the variables x and y fulfill:*

1. *y has no free occurrence in φ, and*
2. *y is substitutable for x within φ,*

then we say that φ, x and y meet the α-β condition, denoted as $C(\varphi, x, y)$.



Lemma 3.1.14. *If $C(\varphi, x, y)$, then $S^y_x S^x_y \varphi = \varphi$.*

3.2 The Axiom System: the Hilbert's System

As for propositional logic, also FOL can be axiomatized.

Definition 3.2.1 (Axioms). *1.* $\varphi \to (\psi \to \varphi)$

2. $(\varphi \to (\psi \to \eta)) \to ((\varphi \to \psi) \to (\varphi \to \eta))$

3. $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$

4. $\forall x\varphi \to S_t^x\varphi$ 替换公理

if t is substitutable for x within φ

5. $\underbrace{\forall x(\varphi \to \psi)} \to (\underbrace{\forall x\varphi} \to \forall x\psi)$ 全称命题

6. $\varphi \to \forall x\varphi$ 全称量词引入

if x is not free in φ

7. $\forall x_1 \ldots \forall x_n\varphi$ 全称概化公式

if φ is an instance of (one of) the above axioms

$\forall x(\varphi \to \psi)$

$\forall x\varphi$

$\varphi \in FOF$

MP Rule: $\dfrac{\varphi \to \psi \quad \varphi}{\psi}$

Definition 3.2.2 (Syntactical Equivalence)**.** *We say φ and ψ are syntactical equivalent iff $\varphi \vdash \psi$ and $\psi \vdash \varphi$.*

Theorem 3.2.3. *(Gen): If x has no free occurrence in Γ, then $\Gamma \vdash \varphi$ implies $\Gamma \vdash \forall x \varphi$.*

Solution. Suppose that $\varphi_0, \varphi_1, \ldots, \varphi_n = \varphi$ is the deductive sequence of φ from Γ.

- If φ_i is an instance of some axiom, then according to (AS7), $\forall x \varphi_i$ is also an axiom.
- If $\varphi_i \in \Gamma$, since x is not free in Γ, we have $\vdash \varphi_i \to \forall x \varphi_i$ according to (AS6). Therefore, we have $\Gamma \vdash \forall x \varphi_i$ in this case.
- If φ_i is obtained by applying (MP) to some φ_j and $\varphi_k = \varphi_j \to \varphi_i$. By induction, we have $\Gamma \vdash \forall x \varphi_j$ and $\Gamma \vdash \forall x (\varphi_j \to \varphi_i)$. With (AS5) and (MP), we also have $\Gamma \vdash \forall x \varphi_i$ in this case.

Thus, we have $\Gamma \vdash \forall x \varphi_n$, i.e., $\Gamma \vdash \forall x \varphi$.

Exercise 3.2.4. *Prove that*

1. $\forall x \forall y \varphi \vdash \forall y \forall x \varphi$,
2. $\exists x \forall y \varphi \vdash \forall y \exists x \varphi$.

Exercise 3.2.5. *Prove that*

1. $\forall x(\varphi \to \psi) \vdash \forall x(\neg\psi \to \neg\varphi)$,
2. $\forall x(\varphi \to \psi) \vdash \exists x\varphi \to \exists x\psi$.

Exercise 3.2.6. *Prove that*

1. *If* $\Gamma \vdash \varphi$ *and* $\Gamma \vdash \neg\psi$*, then* $\Gamma \vdash \neg(\varphi \to \psi)$*,*
2. $\forall x \neg(\varphi \to \psi) \vdash \neg(\varphi \to \exists x \psi)$*.*

Lemma 3.2.7. *(Ren): If* $C(\varphi, x, y)$*, then* $\forall x \varphi$ *and* $\forall y S^x_y \varphi$ *are syntactical equivalent. That is,*

1. $\forall x \varphi \vdash \forall y S^x_y \varphi$*.*
2. $\forall y S^x_y \varphi \vdash \forall x \varphi$*.*

Lemma 3.2.8. *(RS): Let η^{φ}_{ψ} denote the formula obtained by replacing (some or all) φ inside η by ψ.*

If $\varphi \vdash \psi$ and $\psi \vdash \varphi$ then $\eta \vdash \eta^{\varphi}_{\psi}$ and $\eta^{\varphi}_{\psi} \vdash \eta$.

Solution. By induction on the structure of η.

Lemma 3.2.9. *If $C(\varphi, x, y)$ and $\Gamma \vdash \psi$, then $\Gamma \vdash \psi^{\forall x \varphi}_{\forall y S^x_y \varphi}$.*

Solution. An immediate result of (Ren) and (RS).

Theorem 3.2.10. *(GenC) If $\Gamma \vdash S_a^x \varphi$ where a does not occur in $\Gamma \cup \{\varphi\}$, then $\Gamma \vdash \forall x \varphi$.*